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The axiomatics of ordered geometry

I. Ordered incidence spaces

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ABSTRACT

We present a survey of the rich theory of betweenness and separation, from its beginning with Pasch's 1882 *Vorlesungen über neuere Geometrie* to the present.

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Somewhere there is some knowledge; under the stone
A message perhaps; or hidden by strange signs
In places only the stupid visit.

Malcolm Lowry.

Ein Zeichen sind wir, deutungslos,
Schmerzlos sind wir und haben fast
Die Sprache in der Fremde verloren.

Friedrich Hölderlin, Mnemosyne.

1. Introduction

Unlike most concepts of elementary geometry, whose origin is shrouded in the ineluctable fog of all cultural beginnings, the history of the notion of betweenness can be traced back to one person, one book, and one year: Moritz Pasch,¹ *Vorlesungen über neuere Geometrie*, and 1882. The aim of this pioneering book was to provide a solid foundation, very much in the spirit of synthetic geometry, for

¹ For a view of one of Pasch's contemporaries on his life and his work's significance for the axiomatics of geometry, see [325]. While its author attempted a correction of the historical record, this survey can be seen as providing that historical record.

what one could call bounded convex domains living inside projective spaces. Although its approach is philosophically empiricist (see [289, 93, 254, 58, 233: pp. 194–200]), the geometry axiomatized therein being assumed to correspond to aspects of our experience, the extraordinarily precise, strictly deductivist nature of Pasch's geometry ushered in a new era in the axiomatic foundation of geometry. The notion of *betweenness* does not appear in [247] *per se*, as Pasch speaks of 'a point C belonging to segment (*Strecke*) AB instead of the later commonly used expression 'point C lies between points A and B' (although he does refer in [247: §9] to an extension of the concept "between" to the projective plane)². Still in the language of *points belonging to segments* is Peano's [248] axiomatics of ordered geometry, which continues and is influenced by the axiomatic analysis of the notion of order begun by Pasch (see [94]). The presentation in [248] is dressed, unlike the one in [247], in the language of formal logic, and Pasch's philosophically empiricist stance is gone. While in the first ordered geometry Pasch axiomatizes, *points*, *segments*, and the notion of *lying inside a segment* are the only primitive notions, the meaning of "betweenness" is extended in [247: §9] to the notion of *separation* for quadruples of points, to allow for an adequate expression of order in the context of another theory, that of projective geometry. With Hilbert's [117], the next influential book on the foundations of geometry, after the works of Pasch and Peano, order, in the explicit form of "betweenness", becomes only one aspect of Euclidean geometry, alongside incidence, segment-congruence, and angle-congruence.

The aim of this survey is to present the historical development of the axiomatics of geometries of order, in which, much as in Pasch's and Peano's books, no other notion, except on occasion the notion of incidence, is present. This development will be looked at from the point of view of a mathematician, following the thread of ideas and results. To make the narrative mathematically meaningful, there was no way around translating all results into a common language, which happens to be the only possible language, first-order logic, except in the case of Archimedean ordered geometry, in which extensions of first-order logic are needed. This represents a historical distortion on two levels. First, most of the authors did not express their axiom systems in a logical language, and in the case of Sperner's half-ordered geometry, the non-logical language in which he chose to express the axioms was considered a major simplifying feature of the axiomatics, so he most likely would have objected to the present translation. Some, if not most, authors were not that favorably inclined toward formal logic. Second, this translation lets the different axiomatizations communicate, creating the impression that the authors themselves were involved in communication, which was in many instances not the case. Even though the historian will most likely take aim at these distortions, narrating the story from each author or school's point of view, would have significantly lengthened the story and obscured the essential unity of the subject. We have also restricted our survey to purely geometric axiomatizations, excluding the closely related lattice-theoretic approach (see [35–37, 196, 74]), as well as that of abstract convexity, which involves families of subsets, and thus falls outside of our first-order setting.

Regardless of the way one chooses to present the story, an uneasy question raises itself: why should one be interested in the various first-order axiomatizations of geometries in terms of incidence and order alone? To some, this may appear too restrictive from two points of view: the restricted logic (first-order logic, in which quantification is allowed only over individual variables and *not* over sets of variables) and the restricted language, in which no metric notions are allowed, giving rise to what Coxeter [64: p. 176] called "geometry without measurement." That providing a genuine, honest foundation, *ex nihilo* as it were, for geometry or any other area of mathematics can be done *only* in first-order logic had been recognized by most logicians in the 1930s, and by Skolem as early as 1923 (see [89: p. 472], [80]). That geometry does not necessarily need the ability to measure and compare segments or angles had been noticed in the context of projective geometry, the first to systematically avoid metric notions, in effect building up an incidence geometry, being von Staudt in his 1847 book *Geometrie der Lage*. If one allows order as well, then large parts of the geometry of convex sets can be

² The two theories, the one expressed in terms of betweenness and the one expressed in terms of points belonging to segments, are logically equivalent or mutually interpretable, the latter requiring a two-sorted language.

expressed in this restricted language, and many of its traditional theorems remain true. Some of these traditional theorems are Radon's, Carathéodory's, and Helly's theorem, the latter stating that a finite family of at least $d + 1$ convex sets in \mathbb{R}^d has nonempty intersection if every subfamily of $d + 1$ sets has nonempty intersection. In the naive materialist or the bluntly pragmatist view of mathematics, one searches for theorems true in the actual realm of discourse of the dominant paradigm, in the case of geometry some d -dimensional real Euclidean space or the geometry of some manifold, but always endowed with all attributes the space of experience is assumed to have. Of these attributes, continuity in the classical sense is such a strong requirement that its formulation invariably leads to higher-order logics, rendering the axiomatic approach, anchored as it is in first-order logic, at best a harmless distraction. If, however, one takes the Greek view, and asks, as in Pappus's *analysis*, not whether a certain theorem holds, but what does one need to assume for it to hold, then familiarity with the axiomatic enterprise, in particular with the existing axiomatizations of certain fragments of ordered geometry is essential for answering the question regarding the minimal assumptions needed. This act of taking the conditional nature of every mathematical statement seriously has been very eloquently stated in the modern axiomatic context by Hilbert [118]:

Unter der axiomatischen Erforschung einer mathematischen Wahrheit verstehe ich eine Untersuchung, welche nicht dahin zielt, im Zusammenhange mit jener Wahrheit neue oder allgemeinere Sätze zu entdecken, sondern die vielmehr die Stellung jenes Satzes innerhalb des Systems der bekannten Wahrheiten und ihren logischen Zusammenhang in der Weise klarzulegen sucht, daß sich sicher angeben läßt, welche Voraussetzungen zur Begründung jener Wahrheit notwendig und hinreichend sind.

It turns out that very little is needed for the theorems of Radon, Helly, and Carathéodory. Neither the field \mathbb{R} , nor the Euclidean nature of space, nor its dimension are needed in any way for a certain version that implies them all, as shown in [59: pp. 66–67]. Other axiomatic settings in which these three theorems hold have been put forward in [37,48], and in [76] Helly's theorem has been shown to follow from a very general set-theoretic statement.

This kind of questioning ought to be undertaken for any geometric statement that does not involve metric concepts in its statement. Two such examples are the Sylvester–Gallai theorem (“in any finite set of points either all points are collinear or there exists a line containing only two points of the set”) and a problem proposed at the 1979 Putnam competition (“Given n red and n blue points, such that no three are collinear, one can pair each of the red points with a blue point such that the n segments which have these paired points as endpoints are disjoint.”). Easiest to remember among the proofs of the Sylvester–Gallai theorem is one using metric and order concepts by Kelly, yet the theorem remains true in a purely ordered setting. The 1979 Putnam problem used to have only one published proof for more than 20 years, the proof using metric and ordered notions, although its statement is one of ordered geometry. Another concern of Hilbert, that of the *purity of method*, of proving a given statement only with means called for by the statement of the problem, would ask in the Putnam problem case as well for a proof inside a geometry involving only order. That proof was provided in [237], but the quest does not end there. Just as in the case of the Sylvester–Gallai theorem, the proof requires two forms of the Pasch axiom, the inner and the outer form. Are both needed? Could one prove the statement without the outer form?

In a sequel to this paper we plan to present a survey of the axiomatics of richer ordered geometries, in which, as in Hilbert's [117], metric notions are present, but where we focus on the role of order axioms.

2. Order on lines

We start with one-dimensional axioms. By one-dimensional, we mean both axioms in which all the points involved are collinear inside axiom systems without an axiom stating that all points are collinear, and axiom systems for ordered lines.

2.1. One-dimensional order axioms

One-dimensional order axioms have been the most intensely studied axioms in the history of the modern (post-Paschian) axiomatic foundation of geometry.

Starting with [117], order is introduced by means of a ternary relation Z among individual variables to be interpreted as *points*, with $Z(abc)$ to be read as ‘point b lies between a and c ’ (in this variant of the predicate of betweenness, to be referred to as *strict betweenness*, point b must be different from a and from c , and the points a and c are supposed to be distinct). Before Veblen’s [342], the languages in which a variant of ordered geometry was proposed contained, besides Z , a point-line incidence relation. If one wants to avoid adding another sort of individual variables, *lines*, to the language, then one may think of these axiomatizations as being expressed in languages containing Z and λ , where λ is a ternary predicate, with $\lambda(abc)$ to be read as ‘points a, b, c are different and collinear’.

We distinguish here between axiom systems for ordered lines, which include an axiom stating that³

$$\mathbf{A\,2.1.} \quad Z(abc) \vee Z(bca) \vee Z(cab) \vee a = b \vee b = c \vee c = a,$$

and those which do *not* include such an axiom. We are interested first in those in which **A2.1** is not assumed to be holding. The task of finding pure one-dimensional axioms that would completely characterize ordered lines has been raised by Hilbert in the 5th edition of [117], and has been first addressed in [337]. In a first phase, authors did not distinguish between universal order axioms and axioms requiring existential quantification. The universal theory of one-dimensional Z was first singled out as an object of study in [125]. There, the authors first state the axiom of symmetry (**A2.2**), the axiom stating that among three points a, b, c *at most one* of the relations $Z(abc), Z(bca), Z(cab)$ can hold (**A2.3**), and the axiom requiring the points a, b , and c , for which $Z(abc)$ holds, to be different. They also state **A2.1** (an axiom we are excluding for now), and the axioms **A2.5–A2.12** – up to symmetry all universal Z -sentences with four variables that hold in all ordered lines. The object of study of [125] is thus the collection of axioms **A2.1** and

$$\mathbf{A\,2.2.} \quad Z(abc) \rightarrow Z(cba),$$

$$\mathbf{A\,2.3.} \quad Z(abc) \rightarrow \neg Z(acb),$$

$$\mathbf{A\,2.4.} \quad Z(abc) \rightarrow a \neq b \wedge b \neq c \wedge c \neq a,$$

$$\mathbf{A\,2.5.} \quad Z(xab) \wedge Z(aby) \rightarrow Z(xay),$$

$$\mathbf{A\,2.6.} \quad Z(xab) \wedge Z(ayb) \rightarrow Z(xay),$$

$$\mathbf{A\,2.7.} \quad Z(xab) \wedge Z(ayb) \rightarrow Z(xyb),$$

$$\mathbf{A\,2.8.} \quad Z(xab) \wedge Z(ayb) \wedge x \neq y \rightarrow (Z(axy) \vee Z(ayx)),$$

$$\mathbf{A\,2.9.} \quad Z(axb) \wedge Z(ayb) \wedge x \neq y \rightarrow (Z(axy) \vee Z(yxb)),$$

$$\mathbf{A\,2.10.} \quad Z(xab) \wedge Z(yab) \wedge x \neq y \rightarrow (Z(xyb) \vee Z(yxb)),$$

$$\mathbf{A\,2.11.} \quad Z(xab) \wedge Z(yab) \wedge x \neq y \rightarrow (Z(xya) \vee Z(yxa)),$$

$$\mathbf{A\,2.12.} \quad Z(xab) \wedge Z(yab) \wedge x \neq y \rightarrow (Z(xya) \vee Z(yxb)).$$

The analysis of dependencies among these axioms in [125] includes not only all the possible dependencies among **A2.1–A2.12** (i.e. all the pairs (S, i) of subsets S of and elements $i \notin S$ in $\{1, \dots, 12\}$, for which the conjunction of the $A2.j$ with $j \in S$ implies $A2.i$), and there are 71 such dependencies, together with models proving that no other dependency holds among the $A2.i$ s, but also, under the influence of [44] (see also [239]), a mention of the number of times an axiom was used in a particular derivation.

³ Throughout this paper we shall omit the universal quantifiers in front of universal sentences.

From the work done in [125] we can extract several independent subsystems of A2.2–A2.12, in which A2.4 can be replaced by the weaker axiom

A 2.13. $Z(abc) \rightarrow a \neq c$.

They are: {A2.2, A2.3, A2.13, A2.5, A_i, A_j, A_k}, with $i \in \{2.6, 2.7\}$, $j \in \{2.8, 2.9\}$, and $k \in \{2.10, 2.11, 2.12\}$.

A similar analysis, involving 38 five variable universal linear axioms and their relations with A2.5–A2.7, was performed in [257], the first paper, in which the analysis was both restricted to universal sentences and performed under the deliberate exclusion of A2.1. The betweenness relation is no longer Z , but the closely related ternary predicate B , introduced in [165], with $B(abc)$ to be read as ‘ b lies between a and c and may be equal to a or to c ’. The relation B had been previously used by Tarski in a lecture at the University of Warsaw in 1926–1927, and in a book that was to appear in Paris in 1940, but, given the history of that year in Paris, was never printed. It seems that, at least in print, there is only one occurrence in the mathematical literature of B between 1890 and 1942, namely in [281], where the line of inquiry from [165] is carried further. We will refer to axioms written in terms of Z as being consequences of axiom systems written in terms of B and vice versa, given that the two can be easily defined in terms of the other, $B(abc) : \Leftrightarrow Z(abc) \vee a = b \vee b = c$ and $Z(abc) : \Leftrightarrow B(abc) \wedge a \neq b \wedge b \neq c$.

With the collinearity of three different points a , b , and c defined by means of

$$\lambda(abc) : \Leftrightarrow Z(abc) \vee Z(bca) \vee Z(cab), \quad (1)$$

we can define the notion of *line* as a pair of two distinct points (a, b) , and one of *point-line incidence*, by saying that ‘point c is incident with the line determined by a and b ’ if and only if $c = a \vee c = b \vee \lambda(abc)$. Two lines (a, b) and (c, d) are said to be equal if and only if they are incident with the same points. With this definition, we may declare the aim of one-dimensional betweenness axiom systems to be that of providing axioms for ordered spaces in which every line is linearly ordered. The lines geometry is interested in are not only linearly ordered, their order is dense, without first or last element. A first study of these axioms, which are supposed to come after the trivial axioms of point-line incidence, which may be expressed in terms of points as variables and of the ternary relation λ of collinearity (which is supposed to be a primitive notion, i.e. *not* defined by (1)), by means of

A 2.14. $\lambda(xyz) \rightarrow x \neq y$

A 2.15. $\lambda(xyz) \rightarrow \lambda(zyx) \wedge \lambda(yxz)$,

A 2.16. $\lambda(abx) \wedge \lambda(aby) \wedge x \neq y \rightarrow \lambda(axy)$,

was performed in [337]. The aim was to have an axiom system consisting only of *linear* axioms, from which all sentences which hold in dense ordered non-oriented lines without first or last element can be deduced. This was a *purity of the method* issue, given that some of the properties of linear order had been proved in [225, 114], and subsequently included in the 2nd edition and in all future editions of [117], with the use of Pasch axiom, so one was confronted with a situation in which essentially one-dimensional statements, in whose hypothesis there was no mention of non-collinear points, were proved by taking recourse to a two-dimensional axiom, and the assumption of the existence of three non-collinear points. A system of independent axioms for the complete theory \mathcal{D} of dense (non-oriented) linear order without first or last element, put forward in [129: Satz 18], consists of A2.2–A2.4, A2.16’ (by which we mean A2.16, in which λ is considered to be an abbreviation of its definiens in (1), i.e. is an axiom of L_Z), as well as (in A2.19 S_4 stands for the set of all permutations of the set $\{1, 2, 3, 4\}$):

A 2.17. $(\forall ab)(\exists x) [a \neq b \rightarrow Z(axb)]$,

A 2.18. $(\forall ab)(\exists x) [a \neq b \rightarrow Z(abx)]$,

A 2.19. $\lambda(a_1a_2a_3) \wedge \lambda(a_1a_2a_4) \wedge a_3 \neq a_4 \rightarrow [\vee_{\varphi \in S_4} Z(a_{\varphi(1)\varphi(2)\varphi(3)}) \wedge Z(a_{\varphi(1)\varphi(2)\varphi(4)}) \wedge Z(a_{\varphi(1)\varphi(3)\varphi(4)}) \wedge Z(a_{\varphi(2)\varphi(3)\varphi(4)})]$.

Axioms A2.17 and A2.18 stipulate that the order be dense and without first or last element, whereas A2.19 – which states that any four points on a line can be renamed a, b, c, d , such that $Z(abc), Z(abd)$,

$Z(acd)$, and $Z(bcd)$ hold – was taken as an axiom (II 4) in the first edition of [117], to become (both R.L. and E.H.) Moore's theorem in later editions, after it was proved from Hilbert's other incidence and order axioms, including the Pasch axiom, in [114,225] (see also [226]). As shown in [129: Satz 18], if we remove A2.17 and A2.18 from this axiom system, we get an independent axiom system for the theory of all universal linear Z -axioms, i.e. for $\Sigma := \{A2.2\text{--}A2.12\}$.

The first independent axiom system for \mathcal{D} in $L_{\lambda Z}$ was provided in [337]. It consists of $\{A2.2, A2.3, A2.13\text{--}A2.19\}$. Further independent axiom systems for the same theory were presented in [88,234,235]. The one in [88] is expressed in $L_{\lambda Z}$ and consists of $\{A2.16, A2.14, A2.15, A2.20, (1)_{\rightarrow}, A2.17, A2.18, A2.6\}$, where $(1)_{\rightarrow}$ is (1) with \rightarrow substituted for \Leftrightarrow , and A2.20 stands for

A 2.20. $Z(abc) \rightarrow \lambda(abc)$.

Earliest attempts to provide an independent axiom system for this theory go back to [249]; two axioms of that axiom system were proved to be dependent in [282,283].

Various independent axiom systems, from which Σ can be derived, were singled out in [129]. They are: $\{A2.2\text{--}A2.4, A2.16', A2.6, A2.17\}$, $\{A2.2\text{--}A2.4, A2.16', A2.6, A2.18\}$, $\{A2.2\text{--}A2.4, A2.16', A2.6, A2.8, A2.9\}$, $\{A2.2\text{--}A2.4, A2.16', A2.5, A2.6\}$, $\{A2.2\text{--}A2.4, A2.16', A2.7, A2.8, A2.9\}$, $\{A2.2\text{--}A2.4, A2.16', A2.5, A2.11\}$, $\{A2.2\text{--}A2.4, A2.16', A2.19\}$, $\{A2.2\text{--}A2.4, A2.16', A2.6, A2.21_2\}$, $\{A2.2\text{--}A2.4, A2.16', A2.7, A2.11, A2.21_2\}$, where

A 2.21_k. $(\forall abx_1 \dots x_k) (\exists x_{k+1}) [\bigwedge_{i=1}^k \lambda(abx_i) \wedge \bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \rightarrow \bigwedge_{i=1}^k x_{k+1} \neq x_i \wedge \lambda(abx_{k+1})]$.

By using a model with exactly 4 points it is shown that neither A2.17 nor A2.18 can be removed from the axiom systems in which they appear; this being the only independence model (as shown in [301: 2.8] and reproved in [256]), and A2.21₂, stating that there cannot be exactly 4 points on any line, can be substituted for A2.17 or A2.18. It is also shown that none of $\{A2.2\text{--}A2.4, A2.16', A2.6, A2.17, A2.18, X\}$, with $X \in \{A2.5, A2.7, A2.8 \wedge A2.9, A2.11\}$, imply Σ .

Another axiom system, in terms of B , from which the axioms in Σ can be derived, was put forward in [116]. It consists of $\{A2.2_B, A2.7, A2.22, A2.23, A2.24, A2.25\}$, where

A 2.22. $B(aab)$,

A 2.23. $B(axb) \wedge B(abx) \rightarrow x = b$,

A 2.24. $B(axb) \wedge B(axe) \wedge B(bye) \wedge x \neq a \rightarrow (B(abc) \vee B(acb))$,

A 2.25. $B(axb) \wedge B(ayb) \wedge B(cxd) \wedge B(cyd) \wedge x \neq y \rightarrow (B(axe) \vee B(axd))$.

The connection between the ternary notion of betweenness and the binary notion $>$ of linear order (with $a > b$ to be read as ‘point a succeeds point b ’), which axiomatizes the oriented line (an ordered line with a definite sense) was first brought to the fore by Vailati [340] (see also [250]). The axioms for the linear order $>$ (which is denoted by S in [340], and called *serial order* in [121,124]) are: (i) $a > b \vee b > a \vee a = b$, (ii) $a > b \rightarrow b \not> a$, and (iii) $a > b \wedge b > c \rightarrow a > c$. However, the relations $>$ and Z are not synonymous or mutually definable, in the sense that, although Z can be defined in terms of $>$ by $Z(abc) : \Leftrightarrow (a > b > c) \vee (c > b > a)$, one cannot define $>$ in terms of Z . It is rather the pair of orders $\{>, <\}$, called *non-oriented linear order*, and the relation Z , together with three fixed points a, b, c , that are mutually definable, as shown in [129, 254: 9.1, 322]. The n -dimensional generalizations of Vailati's $>$, and with them oriented ordered projective spaces, were axiomatized in [295] (the case $n = 2$ – the corresponding relation $R_2(a_1 a_2 a_3)$ being interpreted as “if a person swims from a_2 to a_3 , then the point a_1 is at his right” – was treated in [294,297]).

All models $\mathfrak{M} = \langle M, Z_M \rangle$ of an axiom system, from which the axioms of Σ can be derived, have the property that, on each of its lines (\mathbf{a}, \mathbf{b}) , with $\mathbf{a} \neq \mathbf{b}$, $\mathbf{a}, \mathbf{b} \in M$, there is a set of opposite order relations $\{<, >\}$, each turning the line (\mathbf{a}, \mathbf{b}) into an ordered set, and such that, for any three points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ on the line (\mathbf{a}, \mathbf{b}) , we have $Z_M(\mathbf{x}\mathbf{y}\mathbf{z})$ if and only if $\mathbf{x} < \mathbf{y} < \mathbf{z}$ or $\mathbf{x} > \mathbf{y} > \mathbf{z}$.

The question regarding the λ -theory of Σ , i.e. the question regarding an axiom system for $\Sigma \cup \{(1)\} \cap L_{\lambda}$, where by L_{λ} we have denoted the language with λ as only primitive notion, has been raised in [51], where it is shown that A2.14, A2.15, and A2.16 imply all five point transitivityes, i.e. all statements in which the conjunction of several collinearities regarding five points, as well as possibly

negated equalities of the points involved, imply a new collinearity. Unsurprisingly, it turns out that $\{A2.14, A2.15, A2.16\}$ is indeed an axiom system for $(\Sigma \cup \{(1)\}) \cap L_\lambda$.

We now turn to the related Z -theory, in which all points are known to be collinear.

2.2. Axiom systems for open non-oriented ordered lines

The first paper to study the complete Z -theory of non-oriented ordered lines was [125]. It contains 11 sets of independent postulates for the Z -theory consisting of all sentences true in all non-oriented ordered lines. These are: $\{A2.1\text{--}A2.4\} \cup X_i$, with $i \in \{1, \dots, 11\}$, where $X_1 = \{A2.5, A2.6\}$, $X_2 = \{A2.5, A2.9\}$, $X_3 = \{A2.5, A2.10\}$, $X_4 = \{A2.5, A2.11\}$, $X_5 = \{A2.5, A2.12\}$, $X_6 = \{A2.6, A2.8\}$, $X_7 = \{A2.6, A2.9\}$, $X_8 = \{A2.7, A2.9\}$, $X_9 = \{A2.7, A2.8, A2.10\}$, $X_{10} = \{A2.7, A2.8, A2.11\}$, $X_{11} = \{A2.7, A2.8, A2.12\}$. In [123], the same theory receives a new independent set of axioms, namely $\{A2.1\text{--}A2.4, A2.26\}$, where

A 2.26. $Z(abc) \rightarrow Z(abx) \vee Z(xbc)$.

All axiom systems $\{A2.1\text{--}A2.4\} \cup X_i$, with one exception, $i = 11$, were shown to be *completely independent* (a notion introduced by E.H. Moore in [227: p. 82], a system Σ of axioms being called *completely independent* if $\Delta \cup \{\neg(\Sigma \setminus \Delta)\}$ is satisfiable for every subsystem Δ of Σ) in [349], and $\{A2.1\text{--}A2.4, A2.26\}$ was shown to be completely independent in [123].

Three independent axiom systems for the same theory, different from those presented in [125, 123], expressed in terms of B , put forward in [216], consist of $\{A2.27, A2.28, A2.29, A2.30\}$, $\{A2.31, A2.29, A2.30\}$, and $\{A2.32, A2.28, A2.33\}$, with

A 2.27. $(\exists z)(\forall abc) [B(abz) \vee B(azb) \vee B(bza) \vee B(baz) \vee B(zab) \vee B(zba)],$

A 2.28. $B(bac) \wedge B(cda) \rightarrow B(dab),$

A 2.29. $B(bac) \wedge B(dba) \rightarrow (B(cad) \vee a = b),$

A 2.30. $B(bac) \wedge B(cad) \wedge B(dab) \rightarrow (a = b \vee a = c \vee a = d),$

A 2.31. $B(abc) \vee B(bca) \vee B(bac),$

A 2.32. $B(abc) \vee B(acb) \vee B(bca) \vee B(bac) \vee B(cab) \vee B(cba),$

A 2.33. $B(cad) \wedge B(cbd) \wedge B(acb) \wedge B(adb) \rightarrow (a = c \vee b = c \vee a = d \vee b = d).$

As shown in [322], in terms of B , a definitionally equivalent theory has $\{A2.34\text{--}A2.36, A2.2_B, A2.6_B\}$ as axiom system, where Ai_B stands for the axiom Ai , in which all occurrences of Z have been replaced with B , and

A 2.34. $B(abc) \vee B(bca) \vee B(cab),$

A 2.35. $B(aba) \rightarrow a = b,$

A 2.36. $B(xab) \wedge B(aby) \wedge a \neq b \rightarrow B(xby).$

As shown in [129: Satz 17c, 256], in the above axiom system, A2.36 can be replaced with A2.37, a variant of A2.21₂, where

A 2.37. $(\forall a_1 a_2 a_3 a_4)(\exists x) [\wedge_{i \neq j} a_i \neq a_j \rightarrow \wedge_{i=1}^4 x \neq a_i].$

Finite models of $\{A2.34\text{--}A2.36, A2.2_B, A2.6_B\}$ can be embedded in the betweenness structure associated to linear trees (connected graphs in which all vertices have degree one or two), where a vertex b is said to be between a and c if b is a vertex of the path connecting a to c , as shown in [68].

That $\{A2.1, A2.2, A2.3, A2.13, A2.6, A2.17, A2.18\}$ forms an independent axiom system for dense non-oriented linear orders without end-elements was first noted in [88].

In [195] it is shown that $\{A2.38, A2.34, A2.35, A2.2_B, A2.6_B, A2.39, A2.40\}$ form an independent axiom system for the first-order L_B -theory of the non-oriented ordered real line. Here

A 2.38. $(\exists ab) a \neq b,$

A 2.39. $(\forall ab)(\exists c) [b \neq c \wedge B(abc)],$

A 2.40. $(\forall ac)(\exists b) [a \neq c \rightarrow a \neq b \wedge b \neq c \wedge B(abc)].$

This theory was among the first to be shown to be decidable, as the decision procedure for dense orders without first or last element from [186] applies to it as well.

In [52], a definitionally equivalent theory received another independent axiomatization in terms of points and the ternary relation H , with $H(abc)$ to be read as ‘ x belongs to the halfline with endpoint a on which b lies’, which can be defined in terms of B by $H(abc) : \Leftrightarrow a \neq b \wedge (B(abc) \vee B(acb))$. The axioms are:

A2.41. $a \neq b \rightarrow H(aba) \wedge H(abb),$

A2.42. $H(abc) \rightarrow a \neq b,$

A2.43. $(\forall p)(\exists qq')(\forall x) \neg H(pqq') \wedge \neg H(pq'q) \wedge (H(pqx) \vee H(pq'x)) \wedge (H(pqx) \wedge H(pq'x) \rightarrow x = p),$

A2.44. $H(abc) \wedge \neg H(bca) \wedge y \neq b \wedge H(bcy) \rightarrow H(axy),$

A2.45. $(\forall ab)(\exists x) [a \neq b \rightarrow H(abc) \wedge H(bac) \wedge x \neq a \wedge x \neq b],$

A2.46. $(\forall abc)(\exists x) [H(abc) \wedge H(bac) \wedge c \neq a \wedge c \neq b \rightarrow ((H(cax) \wedge \neg H(cbx)) \vee (H(cbx) \wedge \neg H(cax)))].$

That the story of betweenness can also be told in terms of the quaternary predicate of *co-directedness* has been first mentioned in [296], and repeated in [129: I.3, 355: I.1, 56]. An independent axiom system in terms of the quaternary relation $/$ of co-directedness (ab/cd being read as ‘the pair of points (a, b) has the same orientation as the pair of points (c, d) ’) for a theory synonymous with {A2.1–A2.4, A2.26} is given in [355: I.1]. Its axioms are

A 2.47. $ab/cd \rightarrow c \neq d,$

A 2.48. $ab/bc \rightarrow ab/ac,$

A 2.49. $ab/xy \wedge cd/xy \rightarrow ab/cd,$

A 2.50. $a \neq b \wedge c \neq d \rightarrow ab/cd \vee ba/cd.$

The synonymy of the two theories (a subject dealt with in [56] as well) amounts to $C_n((A2.1\text{--}A2.4, A2.26, (3)) = C_n A2.47\text{--}A2.50, (2))$, where

$$Z(abc) : \Leftrightarrow ab/bc, \quad (2)$$

$$ab/cd : \Leftrightarrow [(Z(abc) \vee Z(acb) \vee c = b) \wedge Z(acd)] \vee (a = c \wedge (Z(adb) \vee Z(abd) \vee d = b)) \vee (Z(cab) \wedge (Z(cda) \vee Z(cad) \vee a = d)). \quad (3)$$

An independent axiom system in terms of $/$, for a theory synonymous with {A2.38, A2.34, A2.35, A2.2_B, A2.6_B, A2.39, A2.40} was presented in [296]. Non-oriented ordered lines have also received a non-elementary axiomatization in a bi-sorted language in which infinite disjunctions are allowed (i.e. the logic is infinitary or $L_{\omega_1\omega}$), with *points* and *pieces* as individual variables, in [181].

2.3. Metrizable betweenness spaces

The study of *metric betweenness* was started as part of a large project of a foundation of geometry inside the theory of metric spaces by Menger [219: pp. 77–81]. One of his students, Wald, continued this study in [346], and generalized it to metrics with values in the positive cone of ordered Abelian groups in [347]. The underlying idea is that of finding which properties of the betweenness relation B hold in metric spaces, where $B_\varrho(\mathbf{xyz})$ is defined to hold if and only if $\varrho(\mathbf{x}, \mathbf{y}) + \varrho(\mathbf{y}, \mathbf{z}) = \varrho(\mathbf{x}, \mathbf{z})$, where ϱ is the metric of the metric space (taking values, in the generalized version presented in [347], in the positive cone of an Abelian ordered group, and in the previous papers in the positive reals). The subject lay dormant until it was resuscitated in [230], in which structures with both betweenness

and equidistance, defined by a generalized metric, with values in structures more general than ordered Abelian groups, were axiomatized. The subject of pure metric betweenness itself lay dormant even longer, until [217] raised and solved its major problems. It had been noticed already in [346] that the betweenness relation B_ϱ associated with a metric ϱ of a metric space satisfies the axioms A2.2_B, A2.6_B, A2.7_B, A2.35, and A2.22, and that it does not need to satisfy A2.5. Although a complete axiomatization of the theory of metric betweenness had not been the aim of Wald [346], who succeeds in characterizing real metric betweenness spaces topologically, it is quite surprising that the question had not been raised until [217]. We call a betweenness space $\langle M, B \rangle$, where B is a ternary relation on M , *metrizable* if and only if there is an Abelian ordered group G , and a map $\varrho : M \times M \rightarrow G$, satisfying $\varrho(a, b) \geq 0$, $\varrho(a, b) = 0 \Leftrightarrow a = b$, $\varrho(a, b) = \varrho(b, a)$, $\varrho(a, b) + \varrho(b, c) \geq \varrho(a, c)$, such that $B(\mathbf{abc})$ if and only if $B_\varrho(\mathbf{abc})$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in M . It was shown in [217] that this notion is equivalent to the one proposed in [230], where G was a weaker algebraic structure. Several results regarding \mathcal{M} , the first-order theory consisting of all sentences in terms of B true in all metrizable betweenness spaces $\langle M, B \rangle$, were proved in [217]. Every model of \mathcal{M} is a metrizable betweenness space (i.e. the class of all metrizable betweenness spaces is an elementary class), \mathcal{M} has a set of universal *Horn axioms* (universal sentences, for which the conjuncts of the conjunctive normal form consist of disjunctions containing at most one non-negated element), is recursively axiomatizable, but not finitely axiomatizable, is undecidable (although \mathcal{M}_\forall , its universal part, is decidable). The authors of both [217] and [55: pp. 184–189] (which also deals with this problem, without relying on [217]) doubt the possibility of an axiom system consisting of a “reasonable number of readable axiom schemes” or [217: p. 881] or one that would be “both explicit and concise” [55: p. 189]. As shown in [303], where a problem left open in [217] is solved, the theory \mathcal{M}' , the first-order theory consisting of all sentences in terms of B true in all real metrizable betweenness spaces $\langle M, B \rangle$ (i.e. with ϱ taking values in \mathbb{R}) strictly includes \mathcal{M} (i.e. $\mathcal{M}' \setminus \mathcal{M} \neq \emptyset$), and we also have $\mathcal{M}'_\forall = \mathcal{M}'_{\forall\exists} = \mathcal{M}$.

The question left open in [55], whether the Sylvester–Gallai theorem (see [43] for the history of this theorem and for proofs thereof), stated appropriately, remains valid in \mathcal{M} , has been answered affirmatively in [54] (see also [243]).

2.4. Axiom systems for partially ordered lines

Two papers answered in short succession a question raised in [41: Problem 1], for an axiom system in terms of betweenness for partially ordered sets: [12,302]. The problem was revisited in [229,116] (without mentioning [12] or [302]), as well as in [184,335,197], which were aware of [12,302]. Like-minded research was undertaken in [336,7,74,73].

2.5. Cyclic order

The order relation of an oriented circle (clockwise or counterclockwise) will be referred to as *cyclic order*. The first axiom system for it was provided in [121]. What is to be axiomatized are thus the properties the relation C has on the points on the circumference of a circle, where $C(\mathbf{abc})$ holds if and only if the points a, b , and c are distinct and that the arc $a - b - c$, read clockwise, is less than 360° . As shown in [122], completely independent (in the sense of Moore) axiom systems for this theory consist of $\{A2.51, A2.52, A2.3_C, A2.4_C, X\}$, where $X \in \{A2.53, A2.54, A2.55\}$, and

$$\mathbf{A\ 2.51.} \quad C(\mathbf{abc}) \rightarrow C(\mathbf{bca}),$$

$$\mathbf{A\ 2.52.} \quad \wedge_{1 \leq i < j \leq 3} x_i \neq x_j \rightarrow (\vee_{\{i,j,k\}=\{1,2,3\}} C(x_i x_j x_k)),$$

$$\mathbf{A\ 2.53.} \quad C(\mathbf{xab}) \wedge C(\mathbf{ayb}) \rightarrow C(\mathbf{xay}),$$

$$\mathbf{A\ 2.54.} \quad C(\mathbf{xab}) \wedge C(\mathbf{ayb}) \rightarrow C(\mathbf{xyb}),$$

$$\mathbf{A\ 2.55.} \quad C(\mathbf{abc}) \wedge x \neq a \wedge x \neq b \wedge x \neq c \rightarrow (C(\mathbf{abx}) \vee C(\mathbf{xbc})).$$

As shown in [124], for directed lines, the relation C defined in terms of the serial order $<$ by

$$C(abc) :\Leftrightarrow [(a < b \wedge b < c) \vee (b < c \wedge c < a) \vee (c < a \wedge a < b)],$$

satisfies all the axioms it may be expected to satisfy, i.e. {A2.51, A2.52, A2.53, A2.3_C, A2.4_C}.

2.6. Separation of point-pairs on a non-oriented closed line

Just like an open line can be viewed as a directed (oriented) line and axiomatized in terms of the binary relation $>$ or a non-oriented line and axiomatized in terms of the ternary relation Z or B , a closed line can be viewed as directed (oriented), and axiomatized in terms of the ternary relation of cyclic order C , or as non-oriented, and axiomatized in terms of the quaternary relation of separation $//$, with $ab//cd$ to be read as ‘the point-pair (a, b) separates the point-pair (c, d) ’. It was first considered in [247: §9], first axiomatized in [341], and treated at length in [255]. An exhaustive study of all possible dependencies of axioms of separation with four or five variables, in the style of [125], was presented in [126], the last paper belonging to the axiomatic opus of the American postulate theorists, to use an expression coined by [286] (see also [287]). The relevant axioms are:

$$\mathbf{A\ 2.56.} \quad a_1a_2//a_3a_4 \rightarrow \wedge_{1 \leq i < j \leq 4} a_i \neq a_j$$

$$\mathbf{A\ 2.57.} \quad \wedge_{1 \leq i < j \leq 4} a_i \neq a_j \rightarrow (\vee_{\{i,j,k,l\}=\{1,2,3,4\}} a_i a_j // a_k a_l),$$

$$\mathbf{A\ 2.58.} \quad (\exists abcd) ab//cd,$$

$$\mathbf{A\ 2.59.} \quad ac//bd \rightarrow bd//ca,$$

$$\mathbf{A\ 2.60.} \quad ac//bd \rightarrow \neg ad//bc,$$

$$\mathbf{A\ 2.61.} \quad ac//bd \rightarrow db//ca,$$

$$\mathbf{A\ 2.62.} \quad (\exists abcd) ac//bd \wedge db//ca,$$

$$\mathbf{A\ 2.63.} \quad x \neq a \wedge x \neq b \wedge x \neq c \wedge x \neq d \wedge ac//bd \rightarrow (ac//xd \vee ac//bx),$$

$$\mathbf{A\ 2.64.} \quad ax//bc \wedge ac//by \rightarrow ax//by,$$

$$\mathbf{A\ 2.65.} \quad ax//bc \wedge ac//by \rightarrow bc//xy,$$

$$\mathbf{A\ 2.66.} \quad ax//bc \wedge ac//by \rightarrow ac//xy,$$

$$\mathbf{A\ 2.67.} \quad ac//bx \wedge ac//by \rightarrow (ax//by \vee ay//bx),$$

$$\mathbf{A\ 2.68.} \quad ac//bx \wedge ac//by \rightarrow (ax//cy \vee ay//cx),$$

$$\mathbf{A\ 2.69.} \quad ax//bc \wedge ac//by \rightarrow (bx//cy \vee by//cx),$$

$$\mathbf{A\ 2.70.} \quad ax//bc \wedge ac//by \rightarrow (ax//by \vee ay//cx) \wedge (ay//bx \vee ax//cy),$$

$$\mathbf{A\ 2.71.} \quad ax//bc \wedge ac//by \rightarrow (ax//by \vee by//cx) \wedge (ay//bx \vee bx//cy),$$

$$\mathbf{A\ 2.72.} \quad ax//bc \wedge ac//by \rightarrow (ax//cy \vee by//cx) \wedge (ay//cx \vee bx//cy).$$

Huntington and Rosinger [126] established, by means of the most comprehensive deductive analysis ever performed by hand for an axiom system (267 theorems on deducibility and 154 theorems on non-deducibility) that $\{A2.56, A2.57, A2.59, A2.60, A2.61\} \cup X$, with $X \in \{\{A2.63\}, \{A2.65\}, \{A2.66\}, \{A2.64, A_i\}, \text{ with } i \in \{2.67, \dots, 2.72\} \text{ and } \{A2.56, A2.58, A2.62, A2.59, A2.60, A2.63\} \text{ are independent axiom systems from which, when the axiom stating the existence of four different points is added, all of the axioms } A2.56\text{--}A2.72 \text{ can be deduced. Langford [187] established the decidability of the theory of point-pair separation using the above axiom system with } X = \{A2.63\}.$

As shown in [124], the relation $//$, defined in terms of either the betweenness relation Z on a non-oriented ordered line with at least four points on it or in terms of the cyclic order C on closed directed (oriented) lines with at least four points on it by

$$ac//bd :\Leftrightarrow [(Z(abc) \wedge Z(bcd)) \vee (Z(bcd) \wedge Z(cda)) \vee (Z(cda) \wedge Z(dab)) \vee (Z(dab) \wedge Z(abc))], \quad (4)$$

$$ac//bd :\Leftrightarrow (C(abc) \wedge C(cda)) \vee (C(adc) \wedge C(cba)), \quad (5)$$

satisfies all the axioms it could be expected to satisfy, i.e. {A2.58, A2.62, A2.59, A2.60, A2.63}.

One can also define Z in terms of $//$, if $//$ satisfies the axioms ‘There are four points’, A2.56, A2.57, A2.59, A2.60, A2.61, A2.65, and if the language in which the axiom system is expressed contains a constant u as well. Defined by

$$Z(abc) :\Leftrightarrow ac//bu, \quad (6)$$

Z satisfies A2.1 and all the axioms in Σ .

An axiom system for a separation relation, which holds for equal arguments as well (the separation counterpart of B) has been provided in [209], where a notion of convexity was defined as well.

Weaker separation axioms, that are consequences of the axioms of an infinitary axiomatization in the bi-sorted language of *points* and *pieces*, presented in [181], were studied in [128].

2.7. One dimensional ordered projective and affine lines

One dimensional projective geometry has been axiomatized many times and in many guises. The first to show concern for and axiomatize projective lines was Hölder [119], where the language contains one sort of individuals, standing for *points*, and the quaternary relation H , with $H(abcd)$ standing for ‘ d is the harmonic conjugate of c with respect to a and b ’. Lines over skew fields and over algebraic structures more general than skew fields have been axiomatized in [86] in terms of a sexternary relation on points. The group $PGL_2(K)$ of fractional linear (or Möbius) transformations of projective lines over fields K of characteristic $\neq 2$ (transformations $x \mapsto (ax + b)/(cx + d)$ with $a, b, c, d \in K$ and $ad - bc \neq 0$) has been axiomatized in [20]. One axiom of that axiom system has been shown redundant in [198]. The same group, as a group acting on a set (and thus, if the axiom systems were to be expressed in first-order logic, in a bi-sorted language, with *points* (upper-case letters) and *transformations* (lower-case letters) as individual variables, with a binary relation \cdot , whose first argument is a (Möbius) transformation and whose second argument is a point, with $\cdot(g, A)$ (written as $g \cdot A$) to be read as ‘the action of g on A ’, was axiomatized in [334]. The group of affine transformations ($x \mapsto ax + b$, with $a \neq 0, a, b \in K$) of an affine line over a commutative field K was axiomatized in the same bi-sorted language of points and transformations in [334: p. 47, 354]. Affine lines over both skew fields and commutative fields were also axiomatized in [212, 214, 215].

For all of the above axiom systems, one may ask what additional axioms are needed in order to axiomatize ordered projective or affine lines. In the case of projective lines, one needs, in addition to axioms of separation equivalent to {A2.56, A2.57, A2.59, A2.60, A2.61, A2.63}, an axiom stating the existence of four different points, and: (i) an axiom stating the compatibility of the separation relation with H , i.e. that $H(abcd) \rightarrow ab//cd$, in the case of the axiom system in [119], (ii) a similar one for the sexternary operation in [86], (iii) an axiom stating that Möbius transformations preserve the separation relation, i.e. that $AB//CD \rightarrow g \cdot Ag \cdot B//g \cdot Cg \cdot D$. In the case of Bachmann’s [20] axiom system, in which individuals are Möbius transformations, and the only primitive notion is the binary operation of composition of Möbius transformations, the introduction of order has been studied in [314]. In the case of affine lines, one needs to add to the axioms of the axiom systems in [212, 214, 215], axioms of order equivalent to Σ and A2.1, and an axiom of compatibility of Z with the quaternary relation of that language. To the axiom system in [334: p. 47] or [354] one needs to add order axioms equivalent to Σ and A2.1, and an axiom stating that affine transformations preserve the strict betweenness relation, i.e. $Z(ABC) \rightarrow Z(g \cdot Ag \cdot Bg \cdot C)$.

3. Two-dimensional ordered geometries

3.1. Ordered planes

The approach we have followed, of first presenting the one-dimensional axioms before moving on to the two-dimensional ones, has tradition, having been chosen by the founding father himself. §1 of [247] deals quite extensively with the one-dimensional case before introducing axioms for planes and axioms for the order in a plane in §2. We will restrict our attention in this section to the

two-dimensional case, leaving the two-or-higher-dimensional case for the next section. The richest theory one is interested in in this context, to be referred to as *plane ordered geometry* (and whose models will be referred to as *ordered planes*), consists of {A2.2–A2.4, A2.16, A2.6, A2.17, A2.18} as linear axioms, of an axiom stating that there are three non-collinear points, and of a single plane order axiom, which has been always referred to as the *Pasch axiom* (as it was introduced in [247: IV. Kernsatz in §2]), which states that a line that does not pass through any of the vertices of a triangle and intersects one of its sides must intersect another side as well (in this form, the axiom also includes an upper-dimension statement, namely that the dimension is at most two). Formally speaking,

A 3.1. $(\forall abcde)(\exists f)[\neg L(abc) \wedge Z(adb) \wedge \neg L(abe) \wedge \neg L(edc) \rightarrow (Z(afc) \vee Z(bfc)) \wedge L(edf)],$

where L stands for the collinearity relation that also declares collinear three points that are not different, i.e.

$$L(abc) :\Leftrightarrow \lambda(abc) \vee a = b \vee b = c \vee c = a, \quad (7)$$

where λ is either taken to be an independent predicate for which axioms A2.14–A2.16 hold, or a predicate defined in terms of Z by various definitions, such as (1). The only change in the axiom for *ordered planes* needed to allow models of arbitrary dimension ≥ 2 is in the Pasch axiom, which needs to be changed to its so called *outer form* (Axiom VIII in [342: p. 345]), which states that a line that intersects one side and the extension of another side, such that the segment joining the two points of intersection lies *outside* the triangle, must intersect the third side of the triangle as well, i.e.

A 3.2. $(\forall abcde)(\exists f)[\neg L(abc) \wedge Z(abd) \wedge Z(bec) \rightarrow Z(afc) \wedge L(def)].$

Models of the resulting axiom system will be referred to as *ordered spaces*.

We denote by $\Upsilon := \{A2.14, A2.15, A2.16, A3.3\}$ the set of trivial incidence axioms, where A3.3 denotes the lower-dimension axiom

A 3.3. $(\exists abc) \neg L(abc).$

It turns out – as shown, with the aim of eliminating redundant axioms appearing in [117] by R. L. Moore [114] (and [226]) and E. H. Moore [225] (compare also [290,87,234,353]) – that the presence of the Pasch axiom renders several axioms in Σ redundant, so that the axiom system for ordered planes can be stated with very few axioms referring to the order on its lines.

One subject dealt with comprehensively, and without awareness of previous work done in this area, is that of the dependence of A2.2 on some of the other axioms of order. The series of studies was inaugurated by Veblen [344], who proved that, with a different definition of collinearity, introduced in [342] by

$$\lambda_1(abx) :\Leftrightarrow Z(abx) \vee Z(axb) \vee Z(xab), \quad (8)$$

one can prove A2.2 from A2.3, A2.4, as well as the following variant of A2.16, first used in [342]:

A 3.4. $\lambda_1(abx) \wedge \lambda_1(aby) \wedge x \neq y \rightarrow \lambda_1(xya).$

A similar result appeared in [228], where it is shown that A2.3₁ and A3.4 imply a slightly weaker version of A2.2, to which $a \neq b \wedge b \neq c \wedge c \neq a$ has been added to the antecedent, where A2.3₁ is a variant of A2.3 in which the consequent is $\neg Z(bca)$. In a similar vein, with $\lambda_2(xyz)$ defined to hold precisely if at least one of the six permutation of the points x, y, z is in the Z relation, and with A3.4₁ standing for

$$\lambda_2(abx) \wedge \lambda_2(aby) \wedge x \neq y \rightarrow (Z(yxa) \vee Z(xya) \vee Z(xay)),$$

it is shown in [66] that A2.2 (and A2.4) follows from A3.4₁, A3.3 _{λ_2} , A3.2 _{λ_2} , A2.3₁ and A2.18 (where λ_2 indicates that collinearity is expressed by means of λ_2 , i.e. the positive occurrence of L in the conclusion of A3.2 is replaced by λ_2 , whereas the $\neg L(abc)$ occurring in the hypothesis of A3.2 and in A3.3 is replaced by $\neg \lambda_2(abc) \wedge a \neq b \wedge b \neq c \wedge c \neq a$).

Next comes [337], with axioms of point-line incidence and strict betweenness taken from [117] (which can be equivalently expressed with only one sort of variables, for *points*, and with two ternary predicates, λ , for collinearity, and Z), in which it is shown that A2.2 follows from $\Upsilon \cup \{A2.20, A2.18,$

$A2.3, A2.3_1\} \cup X$, where $X \in \{\{A3.1\}, \{A3.1_1\}, \{A3.1_2, A2.3_2\}, \{A3.1_3, A2.3_2\}\}$, by $A2.3_2$ we have denoted the variant of $A2.3$ in which the consequent is $\neg Z(bac)$, by $A3.3$ the lower-dimension axiom, thought of as stated in terms of L , defined as in (7), and by $A3.1_i$ we have denoted $A3.1$, in which the conjunct $(Z(afc) \vee Z(bfc))$ of the consequent is replaced by $(Z(cfa) \vee Z(cfb))$ for $i = 1$, by $(Z(cfa) \vee Z(bfc))$ for $i = 2$, and by $(Z(afc) \vee Z(cfb))$ for $i = 3$. It is also shown that, if $(Z(afc) \vee Z(bfc))$ were to be replaced by $(Z(afc) \vee Z(cfa) \vee Z(bfc) \vee Z(cfb))$ in $A3.1$, an axiom we denote by $A3.1_4$, then $A2.2$ can no longer be deduced from $\{A2.16, A2.14, A2.15, A2.20, A2.18, A2.3, A2.3_1, A2.3_2, A3.3, A3.1_4\}$ (nor can it be deduced from a system to which one adds a host of other congruence axioms for plane Euclidean geometry, as shown in [40]).

That $A2.2$ can be proved from $\mathcal{Y} \cup \{A2.20, A2.18, A2.3, A2.3_1, A2.3_2, A2.4, A3.1_1\}$ was reproved in [348: pp. 154–156]. That $A2.2$ follows from $\mathcal{Y} \cup \{A2.20, A2.18, A2.3, A2.3_1, A2.4, A3.1\}$ and that it also follows from an axiom system containing the strong form of $A3.1_2$, in which the inclusive disjunction $(Z(cfa) \vee Z(bfc))$ is replaced by the either/or disjunction $[(Z(cfa) \wedge \neg Z(bfc)) \vee (Z(bfc) \wedge \neg Z(cfa))]$, was reproved in [353].

The last wave of proofs of $A2.2$, consisting of [71, 194, 213], came in the wake of [42], aiming to show that $A2.2$ is superfluous in their axiom system for plane ordered geometry. The axiom system studied is based, as in [117], on point-line incidence and strict betweenness (but will be expressed here again in terms of λ and Z). The 16 axiom systems from which $A2.2$ can be derived, as shown in [213], are $\mathcal{Y} \cup \{A2.20, A2.4, A2.3_i, A2.18_j, A3.1_k\}$, where $i \in \{0, 1, 2\}, j, k \in \{0, 1\}$, $A2.18_1$ stands for $A2.18$ in which the consequent is $Z(xab)$, and we have denoted by Ai_0 the axiom Ai . The case $i = 2, j = k = 0$ had been treated earlier in [71: I] and the case $i = j = k = 0$ in [194]. By strengthening the Pasch axiom $A3.1$ by dropping the condition $\neg L(abc)$ from its antecedent (to be denoted by $A3.1_s$) and $A2.18$ by adding $x \neq a$ to its consequent (to be denoted by $A2.18_1$), Dubikajtis [71: II] deduced $A2.4$ from $\mathcal{Y} \cup \{A2.20, A2.3_2, A2.3_3, A2.18_1, A3.1_s\}$, and thus showed that this is an axiom system for ordered plane geometry.

That $A2.17$ and $(1)_\rightarrow$ – the former having been a part of axiom II 2, and the latter a part of axiom II 3 in the first six editions of [117] (to become Satz 4 of §3, proved by Wald, in the 7th and all later editions) – follow from $\mathcal{Y} \cup \{A2.20, A2.18, A2.3, A2.3_1, A2.3_2, A2.4, A3.1\}$ has been shown in [337, 45].

The subject of independent axiom systems, in the language of order functions introduced by Sperner [309], for theories definitionally equivalent to $\Sigma \cup \{(1), (7), A3.3, A3.1\}$, i.e. for planes in which the order on a line is not known to be dense and unending, and thus in which it cannot be shown that both of the two sides in which a line divides the plane are non-empty, models of which will be referred to as *ordered planar domains*, was taken up in [169], after having been dealt with in [116] in the language L_B (as well as in [355: Chapter 2] in a language with *points* as the only variables, the quaternary relation $/$ of co-directedness already encountered in 2.2, and the sexternary relation $/_3$ with $abc/_3 a'b'c'$ standing for ‘triangles abc and $a'b'c'$ have the same orientation’ (like-minded axiom systems were offered for n -dimensional spaces, expressed in languages with points as variables and a $2k$ -ary relation $/_k$ for each $1 \leq k \leq n + 1$; the geometry of each of these relations had been studied earlier, in [295: Ch. V])). In terms of λ and Z (or of Z alone, if we think of λ as an abbreviation for its definiens in (1), and no longer list $(1)_\rightarrow$ and $A2.20$ among the axioms), two independent axiom systems for ordered planar domains are $\mathcal{Y} \cup \{A3.5, A2.2, A2.3, A2.4, (1)_\rightarrow, A2.20, A2.5, A2.11, A3.1\}$ and $\mathcal{Y} \cup \{A3.5, A2.2, A2.3, A2.4, (1)_\rightarrow, A2.20, A3.1_\sigma\}$, where by $A3.1_\sigma$ we have denoted the strong form of $A3.1$, in which the inclusive disjunction $(Z(afc) \vee Z(bfc))$ is replaced by the either/or disjunction $[(Z(afc) \wedge \neg Z(bfc)) \vee (Z(bfc) \wedge \neg Z(afc))]$, and by $A3.5$ the axiom stating that on each line there are at least three points, i.e.

A 3.5. $(\forall ab)(\exists c)[a \neq b \rightarrow \lambda(abc)]$,

It was shown in [170: Satz 2] that, although there is neither a density nor an unendingness requirement in the axiom system of planar ordered domains, there are infinitely many points on every line of a planar ordered domain.

Without requiring the existence of enough points on a line, i.e. without an $A3.5$ -like axiom, or for that matter the existence of three non-collinear points, the axiom system in [115], consists of $\{A2.2_B, A2.7_B, A2.22, A2.23, A2.24, A2.25, A3.1_B\}$. One gets an axiom system definitionally equivalent to $\mathcal{Y} \cup \{A3.5, A2.2, A2.3, A2.4, (1)_\rightarrow, A2.20, A2.5, A2.11, A3.1\}$ by adding $A3.3$ and $A3.5$ (with λ replaced by its B -definiens) to it.

That A2.3 becomes superfluous in an axiom system for ordered planar domains in the presence of the strong form of Pasch's axiom A3.1_σ, of axioms excluding the affine planes of order 3 and 5, and of an axiom A3.6, stipulating that, on every line, and for every set of three distinct points $\{x, y, z\}$ on that line, the number of ordered triples $(a, b, c) \in \{x, y, z\}^3$ which are in the relation $Z(abc)$ is a non-zero constant (1, 2, or 3), was shown in [170] (generalizing results in [309, 311, 152, 154]). Thus A2.3 follows from $\Upsilon \cup \{(1)_{\rightarrow}, \text{A2.2, A2.13, A2.5, A2.6, A2.11, A2.20, A2.21}_1, \text{A2.21}_3, \text{A3.1}_{\sigma}, \text{A3.3, A3.6}\}$, which is an axiom system in $L_{\lambda Z}$ for ordered planar domains. Axiom A3.6 can be formulated as (addition being modulo 3 and ${}^{\epsilon}\varphi$ standing for φ if $\epsilon = 1$ and for $\neg\varphi$ if $\epsilon = 0$):

A 3.6. $\lambda(x_1x_2x_3) \wedge \lambda(y_1y_2y_3)$

$$\rightarrow \left[\bigwedge_{k=1}^3 \left(\left(\bigvee_{\substack{\epsilon_j \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = k}} \bigwedge_{i=1}^3 {}^{\epsilon_j}Z(x_i x_{i+1} x_{i+2}) \right) \rightarrow \left(\bigvee_{\substack{\epsilon_j \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = k}} \bigwedge_{i=1}^3 {}^{\epsilon_j}Z(y_i y_{i+1} y_{i+2}) \right) \right) \right].$$

That the axioms of Σ , as well as $(1)_{\rightarrow}$ follow from rather weak plane incidence and order axioms, was pointed out in [31]. The axioms are: Υ , A2.20, A3.5, A2.2, A2.13, $(\exists abc) Z(abc)$, A2.3, A2.3₁, A2.3₂, A2.3₃, A3.1.

Independent axiom systems for plane ordered geometry, with the property that their linear order axioms imply $\Sigma \cup \{\text{A2.17, A2.18}\}$ (as desired by Hilbert in the 6th edition of [117]), were proposed in [129: pp. 47–58].

An interesting chapter in the saga of the axiomatic method in the case of plane ordered geometry is provided by the book [95] of an outsider, the philosopher Moritz Geiger. In it he proposes to build up geometry in a manner that emphasizes the fact that we are forced to choose the axioms that make up the axiom system for geometry if they are to fulfill certain meta-requirements, rather than being free to choose them in an arbitrary manner, restricted only by consistency requirements. To this end the author chooses certain *essential criteria*, from which the individual axioms are derived. Although the enterprise in [95], as the attempt to derive all axioms of geometry from meta-considerations on the nature of the axioms, by some kind of inner necessity, is deeply flawed, as pointed out repeatedly by Kurt Reidemeister in [277–279] (the latter two in a dispute with the author [96], in whose defense his former doctoral student Aron Gurwitsch wrote [109], in a short-lived philosophy journal), there are very fruitful insights that have come out of Geiger's research, applied to the case of plane ordered geometry. The only mathematician who looked positively at [95], and who turned the valuable insights of [95] into new theorems was Forder. In [90, 91] Forder proved several theorems inspired by Geiger's idea and resembling the theorems proved in [95]. Such a theorem, from [91], is: If we ask that no two triangles cut by transversals have contradictory literal symbols (literal symbols are obtained by specifying whether the intersection points the transversal has with the sidelines of the triangle are inside or outside the sides of the triangle, sides ordered according to a clockwise ordering of the vertices), and that there exists at least one figure consisting of a triangle and a transversal meeting all three sidelines, at least two internally, then the outer form of the Pasch axiom, A3.2, in the form presented in [342], is the only one of its kind.

Toward the end of §4 of [117], one finds, without proof, Theorem 9, the Jordan curve theorem for simple polygons, stating the fact that a simple polygon divides the ordered plane into an inner and an outer region, separated by the simple polygon in question. This theorem has the distinction of having been the most often proved result left open in [117], and the most often proved result of ordered geometry, with or without awareness of predecessors.

The first to have provided a proof based on the axioms of plane ordered geometry was Max Dehn in 1899 – at the time a doctoral student of Hilbert – who incidentally was reticent to consider it worthy of publication. We know of his proof thanks to [107, 108]. The first published proof is that in [342: Th. 28]. It was followed by [112, 33: I. Kap. §8] (where it is also shown that the interior of a simple polygon can be decomposed into finitely many triangles with disjoint interiors), [350, 87, 313: §49, 345, 67, 100, 185, 4].

The order topology of ordered planes was studied in [106].

3.2. Ordered affine planes

Plane affine geometry is the L_λ -theory axiomatized by $\mathcal{Y} \cup \{\text{A3.7}\}$, where A3.7 is the Euclidean parallel property, stating, in terms of L , defined in terms of λ by (7), the existence and uniqueness of the parallel line from a point outside of a line to that line, namely

A 3.7. $(\forall abx)(\exists y)(\forall y')(\exists u)[\neg L(abx) \rightarrow y \neq x \wedge (L(xyy') \rightarrow \neg L(aby')) \wedge (\neg L(xyy') \rightarrow L(xy'u) \wedge L(abu))]$.

One can also describe affine planes in terms of the quaternary relation of parallelism \parallel , with $ab \parallel cd$ to mean ‘lines ab and cd are parallel or coincide or $a = b$ or $c = d$ ’, as done in [324] (and simplified in [236]). This relation can be defined in terms of L by

$$ab \parallel cd : \Leftrightarrow (\forall x) (L(abc) \wedge L(abd)) \vee c = d \vee \neg(L(abx) \wedge L(cdx)), \quad (9)$$

and L can be defined in terms of \parallel by $L(abc) : \Leftrightarrow ab \parallel ac$.

That out of the linear order axioms, that are statements regarding more than three variables, one needs only A2.5 to axiomatize ordered affine planes was first noticed in [164] for a restricted class of affine planes (those that can be coordinatized by nearfields), and then for all ordered affine planes in [200]. The axioms are $\mathcal{Y} \cup \{\text{A3.7, A2.2, A2.13, A2.5, A3.8, (1), } (\exists abc) Z(abc)\}$, where

A 3.8. $Z(abc) \wedge L(a'b'c') \wedge \neg L(abb') \wedge aa' \parallel bb' \wedge bb' \parallel cc' \rightarrow Z(a'b'c')$.

As shown in [254] for $i = 1$ – and later in [324] for $i = 1$ and $i = 2$, where the axiom system is in terms of \parallel and B – ordered affine planes can also be axiomatized by $\mathcal{Y} \cup \{\text{A3.7, A2.2, A2.13, A2.6, A3.8}_i, (1), \text{A3.5}\}$, where by A3.8_i we have denoted A3.8, in which, for $i = 1$, $a = a'$ and, for $i = 2$, $b = b'$.

An axiom system for ordered affine planes, which can be easily made elementary using the remarks by Rigby in his MR review, in terms of points and *half-lines* as primitive notions, and the incidence relation, was presented in [72].

As mentioned in [190: p. 338], since the algebraic property of being Euclidean (every element or its additive inverse is a square) of the coordinate field can be expressed in an affine plane in terms of L , the class of Euclidean ordered affine planes can be axiomatized in terms of L alone, Z being an explicitly definable predicate in terms of L . The definition is positive existential in terms of \parallel .

Axiomatizations of ordered affine planes with other primitive notions were put forward in [261].

It was shown in [236: Th. 7] that the classes of ordered translation planes (ordered affine planes with the minor affine Desargues), ordered Desarguesian planes (ordered affine planes with the major affine Desargues axiom), and ordered Pappian planes (ordered affine planes with the affine Pappus axiom) admit axiomatizations in a language with operation symbols, all the axioms being prenex statements containing at most 4 variables.

Matters regarding the effect of the order of an affine plane on the underlying algebraic structure coordinatizing the plane are similar to those of ordered projective planes, and were dealt with in textbook form in [324]. That the skew fields coordinatizing ordered Desarguesian affine planes (or higher-dimensional affine spaces) turn out to be ordered was shown in [14] (see also [288, 199]). An alternate algebraization of both affine and projective ordered spaces can be found in [13] (extended to the half-ordered setting in [298, 299]).

von Plato [258] presents a quantifier-free intuitionist axiomatization of ordered affine planes, and offers in [259] constructive proofs for the Sylvester–Gallai theorem for sets of at most 7 points.

3.3. Ordered projective planes

Plane projective geometry (whose models are called *projective planes*) is the theory axiomatized in terms of λ by $\mathcal{Y} \cup \{\text{A3.5, A3.9}\}$, where A3.9 is the plane projective axiom, stating that any two lines intersect, i.e., with L defined by (7),

A 3.9. $(\forall abcd)(\exists u)[L(abu) \wedge L(cdu)]$.

Projective planes are close relatives of affine planes, in the following precise sense: If we define, inside the theory of affine planes, *projective points* to be pairs of affine points, and define the equality of

projective points by means of a quaternary relation \sim , defined by

$$xy \sim uv : \Leftrightarrow (x = y \wedge u = v \wedge x = u) \vee (x \neq y \wedge u \neq v \wedge xy \parallel uv),$$

and a sexternary predicate of projective collinearity π (with $\pi(x_1 y_1 x_2 y_2 x_3 y_3)$ expressing the fact that the projective points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are collinear) by

$$\pi(x_1 y_1 x_2 y_2 x_3 y_3) : \Leftrightarrow \left[\bigvee_{1 \leq i < j \leq 3} x_i y_i \sim x_j y_j \vee \left(\bigwedge_{k=1}^3 (x_k = y_k \wedge \lambda(x_1 x_2 x_3)) \right) \right. \\ \left. \vee \left(\bigwedge_{k=1}^3 x_k \neq y_k \right) \vee \left(\bigwedge_{k=1}^3 \bigwedge_{i \neq k, j \notin \{i, k\}} x_i = y_i \wedge x_k \neq y_k \wedge x_i x_j \parallel x_k y_k \right) \right],$$

then the points and the collinearity relation thus defined satisfy precisely the axioms of plane projective geometry. A model of this derived theory is referred to as the *projective extension* of the affine plane which is its L_{\parallel} -restriction.

In similar fashion, affine geometry may be defined inside plane projective geometry, if the language of the latter is extended by two constant symbols a_0 and a_1 (standing for two different points), by declaring affine points to be projective points that are not collinear with a_0 and a_1 , and by leaving collinearity unchanged (but restricted to affine points).

Ordered projective planes were first dealt with in [247]. They were dealt with again in [339], a work whose imprecision of language greatly irritated Vahlen's contemporaries Dehn [69] and Veblen [343].

Very close to the axiom system of ordered planes, in that it consists of some universal axioms for the separation relation on a line, an axiom corresponding to the conjunction of A2.17 and A2.18, together with an axiom, A3.13, resembling the strong form of the Pasch axiom A3.1s, is the axiom system for ordered projective planes presented in [313: §64]. It was shown in [6] that axiom II 3 of [313: §64] becomes superfluous if one adds the axioms A2.59 and A2.61. The axiom system, in the simplified version from [6], consists of $\mathcal{Y} \cup \{A3.11, A3.9, A3.10, A2.57', A2.59, A2.60, A2.61, A3.12, A3.13\}$, where by A2.57' we have denoted A2.57 in which the antecedent $\bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j$ has been replaced by $\lambda(a_1 a_2 a_3) \wedge \lambda(a_1 a_2 a_4) \wedge a_3 \neq a_4$ (for which we write in the sequel $\bar{\lambda}(a_1 a_2 a_3 a_4)$), and

$$\mathbf{A\,3.10.} \quad ab // cd \rightarrow \bar{\lambda}(abcd),$$

$$\mathbf{A\,3.11.} \quad (\forall ab)(\exists xy) [a \neq b \rightarrow \bar{\lambda}(abxy)],$$

$$\mathbf{A\,3.12.} \quad (\forall abc)(\exists d_1 d_2) [\lambda(abc) \rightarrow ab // cd_1 \wedge \neg ab // cd_2],$$

$$\mathbf{A\,3.13.} \quad (\forall a_1 a_2 a_3 gh'g'h')(\exists g_1 h_1) \{ \neg L(a_1 a_2 a_3) \wedge (\bigwedge_{i=1}^3 \neg L(hh'a_i) \wedge \neg L(gg'a_i)) \wedge gh // a_2 a_3 \rightarrow [L(gg'g_1) \wedge L(hh'h_1) \wedge ((g_1 h_1 // a_1 a_2 \wedge \neg g_1 h_1 // a_1 a_3) \vee (g_1 h_1 // a_1 a_3 \wedge \neg g_1 h_1 // a_1 a_2))] \}.$$

A similar axiom system was presented in [193].

However, an axiomatization similar to that of ordered affine planes, is the one presented in most papers and books dealing with the subject. Instead of a projective form of the Pasch axiom, these axiom systems contain an axiom stating, much like A3.8, the invariance of the separation relation under perspectivities. This axiom system can be found in [63: Chapter 3, 154, 254: §9, 271: V.1, 160: II.1]. The axiom system presented in [271: V.1] consists of $\mathcal{Y} \cup \{A3.11, A3.9, A3.10, A2.57', A2.59, A2.60, A2.61, A2.65, A3.14\}$, where A3.14 states the invariance of the separation relation under perspectivities, i.e. (with $\lambda_0(abc) : \Leftrightarrow \lambda(abc) \vee b = c$):

$$\mathbf{A\,3.14.} \quad ab // cd \wedge \lambda_0(oaa') \wedge \lambda_0(obb') \wedge \lambda_0(occ') \wedge \lambda_0(odd') \wedge \bar{\lambda}(a'b'c'd') \rightarrow a'b' // c'd'.$$

The topology induced by the separation relation in an ordered projective plane was studied in [351], where the axioms for ordered projective planes are similar to those in [63], i.e. $\mathcal{Y} \cup \{A3.11, A3.9, A3.10, A3.15, A3.16, A3.17, A3.14\}$, where

$$\mathbf{A\,3.15.} \quad \bar{\lambda}(abcd) \rightarrow ab // cd \vee ac // bd \vee be // ad,$$

$$\mathbf{A\,3.16.} \quad ab // cd \rightarrow ab // dc,$$

$$\mathbf{A\,3.17.} \quad ab // cd \wedge bc // de \rightarrow cd // ea.$$

The problem regarding the L_{λ} -theory of ordered projective planes was solved (i) in the Pappian case in [15], (ii) in the Desarguesian case independently in [252, 320], and (iii) for the general case in [147].

The theory contains, besides $\mathcal{Y} \cup \{\text{A3.5, A3.9}\}$, (i) in the Pappian case, beside the projective form of the Pappus axiom, an axiom schema stating, in L_λ , that the sum of non-zero squares is never equal to zero, (ii) in the Desarguesian case, beside the projective form of the Desargues axiom, an axiom schema stating, in L_λ , that the sum of arbitrary products of non-zero squares is never equal to zero, and (iii) in the general case an axiom schema stating, in L_λ , an appropriate generalization of the above condition on sums of products of non-zero squares to ternary fields.

Given the existence of proofs of the Sylvester–Gallai theorem relying solely on order axioms (and thus valid in projective ordered planes), such as those in [62, 64: 12.3], the question arose whether conversely, the validity of Sylvester–Gallai theorem implies the orderability of the projective plane. The answer, given in [138], turned out to be negative. If we denote by \mathcal{P}_2 the L_λ -theory axiomatized by $\mathcal{Y} \cup \{\text{A3.5, A3.9}\}$ and by \mathcal{PO}_2 the $L_{\lambda//}$ -theory of ordered projective planes (i.e. the theory axiomatized by, say, $\mathcal{Y} \cup \{\text{A3.11, A3.9, A3.10, A3.15, A3.16, A3.17, A3.14}\}$), and by SG the set of all Sylvester–Gallai statements, i.e. the set of all statements

$$[\bigwedge_{1 \leq i < j \leq n} \bigvee_{h \notin \{i,j\}} \lambda(a_i a_j a_h)] \rightarrow \bar{\lambda}(a_1 \dots a_n),$$

for $n \geq 4$, then $Cn_{L_\lambda}(\mathcal{PO}_2, SG) \neq Cn(\mathcal{P}_2, SG)$. For minimal systems of axioms from which SG can be proved see [243].⁴

That Desarguesian ordered projective planes can be axiomatized by weakening the Desargues axiom by adding *L(ode)* to the hypothesis of A3.18 (a weak form of the Desargues axiom, also called the *projective minor Desargues axiom*, defining Moufang planes, named after Ruth Moufang, who has introduced them in [231] and studied them intensely in the 1930s) follows from the algebraic results obtained in [306, 46]. There is still no *geometric* deduction of the Desargues axiom from the axioms of ordered projective planes and the projective minor Desargues axiom.

A collineation (bijection preserving both collinearity and non-collinearity) of a projective plane is called a (C, g) -collineation, if it maps every line through the point C and every point on the line g onto itself. A projective plane is called (C, g) -transitive, if for any two of its points P and Q , not on g , and such that C, P , and Q are collinear, there exists a (C, g) -collineation which maps P to Q . As shown in [23], a projective plane is (C, g) -transitive if and only if the (C, g) -Desargues axiom is valid in it (this is the Desargues axiom A3.18, in which the point o is C , and the hypothesis also states that the points d and e lie on the line g).

Projective planes have been divided into classes according to the amount of (C, g) -transitivities they have. The classification was begun by Lenz [188] and refined by Barlotti [32], which is why it is called the Lenz–Barlotti classification. Since each class can be axiomatically fixed by stating that certain (C, g) -Desargues axioms are and certain others are not valid in it, one can ask which of these classes (i.e. which of these axiom systems) can be ordered (remain consistent when extended by the projective order axioms). The most important cases to be settled were those of the non-existence of ordered proper Moufang planes (i.e. of ordered projective planes of Lenz–Barlotti class VII.1) by the results in [306, 46], and that of the orderability of *free* projective planes (introduced in [113]) proved by Jossen [136], a surprising fact from which one infers that the order axioms do not imply by themselves any of the traditional configuration theorems, given that the construction of the free projective plane, starting with three distinct points – a plane of Lenz–Barlotti class I.1 – prevents the occurrence of any unintended collinearity.⁵ The remaining classes were dealt with in [130] (see also [271: p. 270], where the question regarding which Lenz–Barlotti classes contain ordered projective planes was completely answered).

The relationship between ordered projective planes and the algebraic structures coordinatizing them, straightforward in the Pappian case, in which the separation relation induces an order on the coordinatizing field, is significantly more difficult to untangle in the general case, where it was dealt

⁴ The conjecture mentioned in [243: p. 257], that SG ought to remain valid in affine (or projective) planes over fields of characteristic 0, which are not quadratically closed, is false. The fields ought to be 4-formally real fields, i.e. fields in which the sum of at most four non-zero squares is never 0.

⁵ However, the incidence theory of ordered planes is very rich, as shown in [147].

with by Sibylla Crampe [65], who revisits the subject in textbook form in [271], and where the situation is similar: to ordered projective planes correspond ordered ternary fields.

3.3.1. Almost ordered affine planes

An affine plane which can be embedded in an ordered projective plane is called in [139] *almost ordered*. As shown in [139], there are almost ordered affine planes that are not orderable, but all almost ordered translation planes with kernel different from $\text{GF}(2)$ (see [253: 8.2] for a definition of the kernel of a translation plane) are orderable.

3.4. Special classes of ordered planes

3.4.1. Inside open convex subsets of ordered affine planes

Given that the axiom system for ordered planes was set up in [117] with the intention of describing the order structure of Euclidean planes, even though those same axioms may serve as order axioms for hyperbolic geometry as well, one is inclined to believe that ordered planes can be embedded in affine ordered planes. That this is not so, has been first emphasized by Veblen [342: p. 348f]. Veblen constructed an ordered plane whose incidence structure is that of a projective plane, i.e. he showed that the set formed by the axioms for ordered planes and the projective axiom A3.9 is consistent. Veblen's construction has been revisited and generalized to n -dimensional projective space in [178, 36, 160: p. 133f]. Veblen [342: Th. 38] also showed that, in the presence of the second-order continuity axiom (implying that lines have the second-order betweenness properties of the ordered set of real numbers), such models can no longer exist, i.e. that one can prove $\neg\text{A3.9}$. Szczerba [317] provided models for ordered planes, in which the projective form of the Desargues axiom holds, with projective incidence structure with any field that is not isomorphic to \mathbb{R} as coordinate field, showing that the full strength of the second-order Cantor–Dedekind continuity axiom is needed to prove $\neg\text{A3.9}$, even if the axiom system for ordered planes is enlarged by the Desargues axiom (in fact, even if enlarged by the projective forms of the Desargues and the Pappus axioms). The Desargues axiom states that

A 3.18. $L(oa'a') \wedge L(obb') \wedge L(occ') \wedge L(abd) \wedge L(a'b'd) \wedge L(ace) \wedge L(a'c'e) \wedge L(bcf) \wedge L(b'c'f) \wedge \neg L(oab) \wedge \neg L(oac) \wedge \neg L(abc) \wedge a \neq a' \wedge b \neq b' \wedge c \neq c' \wedge a' \neq o \wedge b' \neq o \wedge c' \neq o \rightarrow L(def)$,

whereas the Pappus axiom states that if a and b are different lines containing points a_1, a_2, a_3 and b_1, b_2, b_3 respectively, with $a_i \neq b_j$ for all $i, j \in \{1, 2, 3\}$ and $a_i \neq a_j, b_i \neq b_j$ for $i \neq j$, and m lies on the lines a_1b_2 and a_2b_1 , n lies on the lines a_1b_3 and a_3b_1 , and p lies on the lines a_2b_3 and a_3b_2 , then m, n , and p are collinear. The elementary continuity axiom, in the form that can be found in [319, 293: p. 416], states that, for all formulas $\alpha(x)$ of L_B , in which the variables a_1, \dots, a_n, x (with $n \in \mathbb{N}$), but not b, c, y occur free, and for all formulas $\beta(y)$ of L_B , in which the variables a_1, \dots, a_n, y , but not b, c, x occur free, we have

A 3.19. $(\forall a_1 \dots a_n) \{ (\exists b)(\forall xy) [\alpha(x) \wedge \beta(y) \rightarrow B(bxy)] \rightarrow (\exists c)(\forall xy) [\alpha(x) \wedge \beta(y) \rightarrow B(xcy)] \}$.

It is most remarkable that, if the Desargues axiom A3.18 is modified by replacing L wherever it appears *un-negated* in the exact syntactic form in which we have expressed it above, by B , an axiom we shall refer to as A3.18_o, then there are no models of ordered planes satisfying A3.18_o, A3.19, and A3.9, i.e. it is not possible for these models to have a projective incidence structure, given that all models of ordered planes satisfying A3.18_o and A3.19 are, as shown in [319], isomorphic to open convex subsets of some affine planes over ordered real-closed fields. The main problem for this theory, called *general affine geometry* or \mathcal{GA}_2 , is that of characterizing its models, given that it is *not* true that all convex subsets of affine planes over real closed fields are models of \mathcal{GA}_2 . In fact, for any ordered real closed field F , not isomorphic to \mathbb{R} , there are open convex subsets of the ordered affine plane over F , which are not models of \mathcal{GA}_2 . However, all open convex subsets of the ordered affine plane over \mathbb{R} are models of \mathcal{GA}_2 . Describing those open convex subsets of an ordered affine plane over a real closed field that can serve as universes for models of \mathcal{GA}_2 , a problem raised in [319], turns out to be impossible, given that their structure cannot be captured by a statement in first-order logic (not even in weak second order logic, that allows quantification over finite sets), as shown in [267]. A wealth of metamathematical results regarding \mathcal{GA}_2 (as well as hints on how to extend some results

to the n -dimensional case) has been obtained in [319,268,267] (and summarized in [293: II.7], where additional results, communicated by Szczerba and Prestel, are mentioned). Among these: \mathcal{GA}_2 is not finitely axiomatizable, is hereditarily undecidable, it is a proper subtheory of the L_B -theory \mathcal{GA}_2^+ of the class of all open convex subsets of the ordered affine plane over \mathbb{R} , since \mathcal{GA}_2^+ is not recursively axiomatizable. The theories \mathcal{GA}_2^+ and \mathcal{GA}_2 cannot be distinguished by means of sentences in prenex form with prefixes of type $\forall\exists, \exists^1\forall^2\exists$ and $\forall^2\exists\forall$ (here Q^n indicates that the quantifier Q can appear at most n times; where the superscript n is missing, Q is allowed to occur any number of times). In \mathcal{GA}_2 , from every point to every ray there is a limiting parallel ray (see [293: 7.10(i)]), and one may ask what properties the *oriented parallelism* relation $\uparrow\uparrow$ (with $ab\uparrow\uparrow cd$ to be read as ‘the rays ab and cd are parallel, or $a = b$ or $c = d$ ’ or equivalently ‘ ab is parallel and has the same orientation as cd , or $a = b$ or $c = d$ ’) that one can define in \mathcal{GA}_2 , has. A first step in the direction of answering this question was taken by Grochowska–Prażmowska [103], where a (dimension-free, i.e. without an upper-dimension axiom) axiom system for an ordered geometry based on the quaternary relation of oriented parallelism $\uparrow\uparrow$ is provided. Its axioms are A2.38 and:

A3.20. $ab\uparrow\uparrow cc$,

A3.21. $ab\uparrow\uparrow ba \rightarrow a = b$,

A 3.22. $a \neq b \wedge ab\uparrow\uparrow pq \wedge ab\uparrow\uparrow rs \rightarrow pq\uparrow\uparrow rs$,

A3.23. $ab\uparrow\uparrow bc \rightarrow cb\uparrow\uparrow ba$,

A3.24. $ab\uparrow\uparrow bc \rightarrow ab\uparrow\uparrow ac$,

A3.25. $ab\uparrow\uparrow ac \rightarrow ab\uparrow\uparrow bc \vee ac\uparrow\uparrow cb$

A3.26. $(\exists abcd)[\neg ab\uparrow\uparrow cd \wedge \neg ab\uparrow\uparrow dc]$,

A3.27. $(\forall abc)(\exists d)[ab\uparrow\uparrow cd \wedge ac\uparrow\uparrow bd \wedge d \neq b]$,

A3.28. $(\forall abcp)(\exists d)[b \neq p \wedge bp\uparrow\uparrow pc \rightarrow ap\uparrow\uparrow pd \wedge ab\uparrow\uparrow cd]$.

Several forms of the Pasch axiom were shown to hold in this theory in [103]. If one adds to this axiom system the axiom

A3.29. $ab\uparrow\uparrow cd \rightarrow ba\uparrow\uparrow dc$,

then the resulting axiom system is definitionally equivalent to (dimension-free) ordered affine geometry (as shown in [101]). That this theory is also logically equivalent to the theory of groups of positive dilatations, in which translations are singled out, was shown in [102], where one also finds an axiom system for the latter theory, with two sorts of individual variables, for *positive dilatations* and for *translations*.

We mentioned earlier that Desarguesian ordered planes (i.e. ordered planes satisfying A3.18) need not be embeddable in affine ordered planes, not even if they satisfy the axiom schema A3.19. What needs to be added to the axiom system of Desarguesian ordered planes is an axiom stating, in essence, that, for every model, there exists a line in its projective extension which lies outside the model’s universe. In precise language,

A 3.30. $(\exists abcd)(\forall xyzvws)[L(axb) \wedge L(byc) \wedge L(czd) \wedge L(dua) \wedge L(avc) \wedge L(xvz) \wedge L(yvu) \wedge L(xus) \wedge L(yzs) \rightarrow x = u \vee y = z]$.

This axiom is part of the definition of *convexity*, as defined by Steinitz [312], for sets in ordered projective planes. The definition for a subset S of an ordered projective plane \mathfrak{P} to be convex consists of two requirements: that (i) any two points in S can be joined by a segment which is contained in S , and that (ii) S be disjoint from some line of \mathfrak{P} . Two distinct points a and b of \mathfrak{P} determine, by A3.12, two open segments: if c is another point in the line ab , then one segment consists of all x for which $ab//cx$, and the other segment consists of all points for which $\neg ab//cx$. It was shown in [105] that a set S in \mathfrak{P} is

convex if and only if it contains no line of \mathfrak{P} , i.e. conditions (i) and (ii) of the definition can be replaced by a strengthened form of (i), asking that any two points in S can be joined *uniquely* by a segment which is contained in S . Subsets S of \mathfrak{P} satisfying only (i) are called *semi-convex*. Szczerba [316] showed that all models of the theory of Desarguesian ordered planes, to which we add the definition of the separation relation (4), are isomorphic to the ordered geometry of semi-convex subsets of ordered projective planes coordinatized by skew fields. If we add the projective form of Pappus's axiom to the axiom system of Desarguesian ordered planes, the skew fields in their representation theorem become commutative fields. No proof that the projective form of the Pappus axiom implies the projective form of the Desargues axiom based on the axioms for ordered planes exists in print. Szczerba communicated to the author having once had such a proof.

3.4.2. Hyperbolic geometry

A special case of ordered planes, is that in which the universe of the ordered plane is the interior of an ellipse in an ordered Pappian affine plane whose coordinate field is Euclidean (every positive element has a square root). Such ordered planes may be called *hyperbolic planes*, given that the metric notions usually associated with hyperbolic geometry, such as segment congruence or orthogonality, can be defined in terms of collinearity. This discovery, together with that of the fact that B itself can be defined in terms of λ in hyperbolic geometry, has lead Menger [220] to conclude that plane hyperbolic geometry (in fact, that dimension-free hyperbolic geometry, as shown later in [293]) (see also [356–358]) is definitionally equivalent with a theory expressed in L_λ . The combined effort of Menger [220–223], and his students, Abbott [1–3], Jenks [133], de Baggis [24,25] (see also the survey [8]), and finally Skala [305] (where the axiom system stated below was proposed) led to a particularly appealing axiom system.

To shorten the statement of some of the axioms we define *lines* as pairs of distinct points $\langle a, b \rangle$, and say that a point p is incident with (or on) line $\langle a, b \rangle$ if $L(abp)$, and that two lines are equal if they are incident with the same points, as well as: (1) the notion of strict betweenness Z , with $Z(abc)$ defined by means of: 'the points a, b , and c are three distinct collinear points and every line through b intersects at least one line of each pair of intersecting lines which pass through a and c '; (2) the notions of ray and segment in the usual way, i.e. a point x is on (incident with) a ray \overrightarrow{ab} (with $a \neq b$) if and only if $x = a$ or $x = b$ or $Z(axb)$ or $Z(abx)$, and a point x is incident with the segment \overline{ab} if and only if $x = a$ or $x = b$ or $Z(axb)$; (3) the notion of ray parallelism, for two rays \overrightarrow{ab} and \overrightarrow{cd} not part of the same line by the condition that every line that meets one of the two rays meets the other ray or the segment ac ; two lines or a line and a ray are said to be parallel if they contain parallel rays; (4) the notion of a rimpoint as a pair $(\langle a, a' \rangle, \langle b, b' \rangle)$ of parallel lines, which is said to be incident with a line $\langle l, l' \rangle$ if $\langle l, l' \rangle = \langle a, a' \rangle$, $\langle l, l' \rangle = \langle b, b' \rangle$ or $\langle l, l' \rangle$ is parallel to both $\langle a, a' \rangle$ and $\langle b, b' \rangle$, and there exists a line that intersects $\langle a, a' \rangle$, $\langle b, b' \rangle$, and $\langle l, l' \rangle$; a rimpoint $(\langle a, a' \rangle, \langle b, b' \rangle)$ is identical with a rimpoint $(\langle c, c' \rangle, \langle d, d' \rangle)$ if both $\langle c, c' \rangle$ and $\langle d, d' \rangle$ are incident with $(\langle a, a' \rangle, \langle b, b' \rangle)$. Rimpoints will be denoted in the sequel by capital Greek characters. If Π_1 and Π_2 are rimpoints, then $\Pi_1 \Pi_2$ denotes the line $\langle l, l' \rangle$ incident with both Π_1 and Π_2 , and $\Pi_1 p$ denotes the line $\langle l, l' \rangle$ incident with Π_1 and p .

The axiom system, that we think of as expressed in L_λ , but which we present in informal language, its formalization being straightforward, consists of \mathcal{Y} , together with:

A 3.31. $(\exists abc) \lambda(abc)$,

A 3.32. Of three collinear points, at least one has the property that every line through it intersects at least one of each pair of intersecting lines through the other two.

A 3.33. If p is not on $\langle a, a' \rangle$, then there exist two distinct lines through p not meeting $\langle a, a' \rangle$ and such that each line meeting $\langle a, a' \rangle$ meets at least one of those two lines.

A 3.34. Any two non-collinear rays have a common parallel line.

A 3.35 (*Pascal's theorem on hexagons inscribed in conics*). If Π_i ($i = 1, \dots, 6$) are rimpoints and m, n, p are the intersection points of the lines $\Pi_1 \Pi_2$ and $\Pi_4 \Pi_5$, $\Pi_2 \Pi_3$ and $\Pi_5 \Pi_6$, $\Pi_3 \Pi_4$ and $\Pi_6 \Pi_1$, then m, n , and p are collinear,

as well as the projective forms of the Desargues and Pappus axioms formulated for points only (Desargues is probably superfluous, according to Szczerba's communication mentioned above). By

changing the definition of ray-line parallelism, as indicated in [240], this axiom system can be made to consist of $\forall\exists\forall\exists$ -sentences, and there is, as shown in [240], no axiom system in L_λ or in L_Z of lower quantifier complexity.

It follows from [24] that the theory axiomatized by the axiom system formed of γ , A3.31–A3.34, together with the projective form of the Desargues axiom A3.18 is definitionally equivalent to plane ordered geometry, to which the axioms A3.33 and A3.34 have been added, and thus are special cases of Desarguesian ordered geometry in which the open convex set S in the ordered projective plane \mathfrak{P} , which is the universe of a model of this theory, is not only included in $\mathfrak{P} \setminus g$, where g is a line in \mathfrak{P} , but also that, for every line l that S contains, there is a projective line \bar{l} , with $l \subset \bar{l}$ and such that \bar{l} intersects ∂S in two points (here ∂S stands for the boundary of S in \mathfrak{P} seen as a topological space with the topology induced by its order). The special case in which the subset S of \mathfrak{P} is a triangle has been dealt with in [251, 115], and has received a $\forall\exists\forall\exists$ -axiomatization (which, just like the one for hyperbolic planes, is optimal as far as quantifier complexity is concerned) in L_λ in [241].

An $\forall\exists\forall\exists$ -axiomatization in L_λ exists for Klingenberg's [167] (see also [21: Chapter V]) generalized hyperbolic planes, whose universes are the interiors of ellipses in affine planes over arbitrary ordered fields (they are thus more general than the models of hyperbolic geometry, given that the ordered field's Euclidean condition is dropped, and thus lines in these models do not necessarily have 'ends', i.e. do not need to intersect the absolute) as well (see [240]).

An axiomatization in terms of the relation of oriented parallelism $\uparrow \downarrow$ for the degenerate hyperbolic plane, which consists of the strip between two parallel lines in an ordered affine plane was provided in [260]. An axiom system for the dual hyperbolic plane can be found in [224].

3.5. Projective geometry is all geometry

In 1859, Cayley [53: p. 90 (592)] wrote that “descriptive geometry is *all* geometry and reciprocally”, and by “descriptive” he meant what we mean by “projective”. It turns out that all two-dimensional structures we have encountered can indeed be seen to live inside ordered projective planes. Although foreseen by Pasch [247] for the three-dimensional ordered spaces, this insight was gained only gradually and with considerable effort.

What this embedding problem amounts to, as seen from a language-immanent point of view, can be best understood when one thinks of the geometry to be embedded as being expressed in terms of the incidence and separation relations. Since \parallel can be defined by means of (4) in terms of Z , it is meaningful to look at the L_{\parallel} -theory of a theory \mathcal{T} expressed in L_Z . However, the two theories are *not* definitionally equivalent (as Z cannot be, in general, defined in terms of \parallel). The axiom system closest in spirit to one in terms of incidence and of \parallel is the one put forward in [352]. There, the incidence axioms are those of Hilbert [117], appropriately modified to allow spaces of arbitrary dimensions ≥ 3 , and the primitive notions are *point*, *line*, *plane*, with three kinds of incidences: point-line, point-plane, and line-plane. As with all incidence geometries, the notion of a *flat* is introduced, and its definition is not one of first-order logic. The notion then shows up in the axioms of separation, as in expressions of the kind ‘a line g passing through a point P and lying in the plane determined by point P and line l , which does not go through P ’. Separation itself is not a quaternary relation among points in [352], but one among lines, by means of which our \parallel can be defined. In the presence of order, the shortcoming just mentioned, that the notion of plane spanned by a non-incident point-line pair cannot be expressed in first-order logic, can be removed, as first shown in the case of ordered spaces by Veblen [342]. In a similar manner, planes spanned by a non-incident point-line pair can be defined in first-order logic in terms of the separation relation among lines used in [352], and thus the entire axiom system can be turned into one in first-order logic. Wyler presents a definition for Z inside his axiom system (with $Z(abc)$ holding if and only if a , b , and c are three different points, incident with a line l , and there is a point p , which is not on l , and a line t , in the plane spanned by p and l , but not intersecting l , such that the pair of lines (pa, pb) separates the pair of lines (pc, t)), and if there are three points a , b , and c , such that $Z(abc)$ (which is not always the case, as the incidence and separation axioms do not exclude the projective case, in which there can be no such triple of collinear points, as all lines t in the plane spanned by p and l intersect l), then Z satisfies all the axioms for ordered spaces. In particular, ordered affine planes (or ordered affine spaces of arbitrary dimension ≥ 2) can be axiomatized in terms of

separation (i.e. with points as individual variables and \parallel_a as primitive notion (as L can be defined in terms of \parallel_a)), and we may ask whether there is a separation relation \parallel_p defined on its projective extension which coincides with \parallel_a on the points of the affine plane, and if so, whether \parallel_p is unique. The positive answer to his question can be easily checked in case the affine plane is Desarguesian (or in the case of affine spaces of dimension ≥ 3), but the general case, where the answer is positive as well, is harder to establish, being first proved in [254: §9.2], then in [65] (see also [271: Satz 8]). In the more general setting, presented in Section 5, of Sperner's half-ordered planes, introduced in [309], the extendability problem was treated in [311, 155, 135, 179]. A corollary of the result in [65] (mentioned in [271: Korollar 9]) is that in the axiom system $\mathcal{V} \cup \{A3.11, A3.9, A3.10, A2.57', A2.59, A2.60, A2.61, A2.65, A3.14\}$ of ordered projective planes, A3.14 can be weakened, by changing its universal prefix to $(\exists a_0 a_1)(\forall aa'bb'cc'dd'o)$ and adding $L(a_0a_1o)$ to its hypothesis.

In the case of ordered planes, the first to show that an ordered plane can be embedded in a projective ordered plane, such that the restriction of the projective separation relation coincides with the separation relation of the ordered plane obtained from its betweenness relation by means of (4), was Ruth Moufang [231]. She showed this only for planes which satisfy the projective minor Desargues axiom (whenever all points involved belong to the ordered plane), and proved that the ordered projective plane in which it can be embedded satisfies the projective minor Desargues axiom as well. Sperner [308] showed that the same result holds for ordered planes in which the projective Desargues axiom holds, and proved that the projective plane in which it can be embedded satisfies the projective Desargues axiom as well. It follows from the results of [306, 46] that the result from [231] implies the embedding result from [308]. It took the groundbreaking paper of Joussem [136] on the orderability of free extensions of incidence structures satisfying certain properties to show that the embedding in a projective ordered plane is possible for an arbitrary ordered plane. The results from [308, 252, 320] also solve the problem regarding the L_λ -theory of ordered planes in which the projective form of the Desargues axiom holds, and those of [136, 147] the problem regarding the L_λ -theory of all ordered planes. These theories are recursively axiomatizable, but the axiom systems one gets by using the fact that arbitrary ordered planes have projective ordered planes obtained as free extensions, as in [136], or that Desarguesian ordered planes can be extended to Desarguesian projective ordered planes, as in [308], contain infinitely many axioms whose description is, to say the least, uncomfortably complex.

3.6. Double elliptic geometry

The order on a (2-dimensional) sphere can be described in terms of Z by having as one of the axioms $(\forall a)(\exists b)(\forall c) \neg Z(abc)$, stating that every point a has an antipode b , and that the line determined by a and b consists of the points lying between these two points. This approach was taken in an axiomatization of 3-dimensional double elliptic geometry in terms of *points* and Z alone in [166]. Except for the last axiom (Axiom X), which is a second-order continuity axiom, the axioms are first-order statements, and, by suitably changing the upper-dimension axiom (Axiom IX), the first-order axiom system can be turned into one for elementary double-elliptic geometry of any dimension ≥ 2 .

4. Higher dimensions

Higher-dimensional ordered affine and projective spaces are easily axiomatized by adding to the axioms for higher-dimensional affine or projective spaces order or separation axioms. In the affine case, we may choose to add to the axioms for dimension-free (at least three dimensional) affine spaces in [183] axioms for dense unending non-oriented linear orders and A3.8. An axiom system for dimension-free Pappian ordered affine spaces, in which the two-dimensional case is included, was put forward in [104]. In the projective case, we may choose to add to the axioms of dimension-free (at least three dimensional) projective spaces from [189] the linear separation axioms and the axiom requiring the invariance of the separation relation under projectivities A3.14. Given that the Desargues axiom holds in such models, all models for these theories are isomorphic to (affine or projective) spaces constructed over ordered skew fields. The relationship between an ordered projective space and its associated ordered Grassmann space is the subject of [39].

An axiomatics for 3-dimensional ordered affine spaces based on maps, called *3-orders*, assigning to ordered quadruples of points one of the values $-1, 0, 1$, which can be adapted to one for n -dimensional

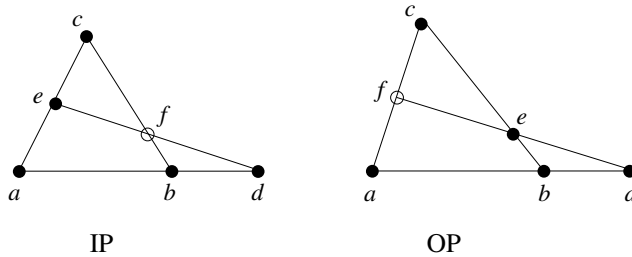


Fig. 1. The inner and the outer form of the Pasch axiom.

ordered spaces for any $n \geq 2$, has been provided in [110] (with independence models in [111]). The author was apparently unaware that the set-up, in particular the notion of 3-orders, differs from that of [140] (which originates in the work of Sperner [309], continued in [19,192,92,98] (see also [120,162]), where Sperner's order functions are replaced by orientation functions comparing the orientation of two simplices) only in that the 3-orders are defined on all quadruples of points and are allowed to take the value 0, allowing the definition of the incidence-theoretic notions of the affine space, whereas the *area functions* in [140] are defined only on simplices, i.e. for points that do not lie in the same hyperplane, and cannot take a value without a multiplicative inverse. The axioms can be expressed in terms of *points* and, for the 2-dimensional case, two ternary relations σ_1 and σ_{-1} , with $\sigma_i(abu)$ standing for a generalization of the notion of 'the sign of the determinant $|a - u \ b - u|$ is that of i '. For the n -dimensional case, one needs two analogous $(n+1)$ -ary relation symbols.

The class C of all structures $\mathfrak{M} = \langle M, B_M \rangle$ which are embeddable into some ordered Pappian affine space of unspecified dimension turns out, as shown in [218], to be an elementary class (i.e. the class of models of all L_B sentences true in C coincides with C), recursively, but not finitely, axiomatizable, and such that its universal theory is decidable. With C_0 standing for the class of all $\mathfrak{M} = \langle M, B_M \rangle$ which are embeddable into some affine space of unspecified dimension over \mathbb{R} , it was shown in [304], that C is not the least elementary class containing C_0 , but it is both the least universal class and the least $\forall\exists$ class containing C_0 .

Higher-dimensional ordered spaces were introduced as early as [247,248,342], where only the three-dimensional case is specifically mentioned, but where the methods can be readily generalized to the n -dimensional case. This task was taken up in [285]. As mentioned earlier, the only axiom forcing two-dimensionality in the axiom system for ordered planes is the Pasch axiom in the form A3.1. The task of rephrasing it, such as to retain its function in providing a meaningful definition of halfplanes while removing its two-dimensional implications, was taken up by Peano [248] (see also [280]), where two versions of the Pasch axiom were introduced, to be called the *outer* and the *inner* form of the Pasch axiom (in [248] Axioms XIII and XIV, see Fig. 1). The outer form, as introduced by Peano, was stronger than A3.2, as it required $Z(def)$ and not just $L(def)$ in its conclusion. That it can be weakened to A3.2, in the sense that the stronger form of Peano can be proved from it and from Σ , has been shown by Veblen [342: Th. 7]. There we also find the proof that the inner form of the Pasch axiom (which still figured, together with the outer form, as an axiom in [248]) is superfluous, as it can be derived from \mathcal{D} and A3.2 (or from A3.5 and A3.2, as shown in [246]). The inner form of the Pasch axiom states that (Fig. 1)

A 4.1. $(\forall abcde)(\exists f) \neg L(abc) \wedge Z(abd) \wedge Z(aec) \rightarrow Z(bfc) \wedge Z(dfe)$.

In [342: p. 351] it is erroneously claimed that A3.2 (to be referred to as **OP**) can be deduced from A4.1 (to be referred to as **IP**), A3.3, and \mathcal{D} . That **IP** is strictly weaker than **OP**, i.e. that \mathcal{D} , A3.3, and **IP** do not imply **OP** has been shown syntactically, by analyzing possible proofs, in [244]. It was also shown in the same paper that one can split **OP** into the axioms **IP** and **WP** (i.e. one can replace **OP** by these two axioms, each weaker than **OP**), the *weak* Pasch axiom (introduced by Szmielew in [323]), which is A3.2, in whose conclusion $Z(afc)$ is replaced by $L(afc)$.⁶ It is not known

⁶ The proof that **OP** can be split into **IP** and **WP** can also be found among the papers in the *Nachlass* of Martin Schröder at the University of Duisburg.

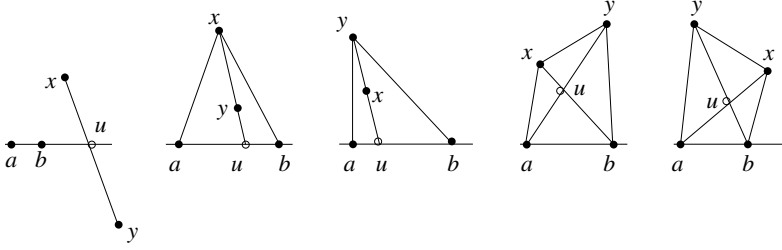


Fig. 2. Axiom A 4.2.

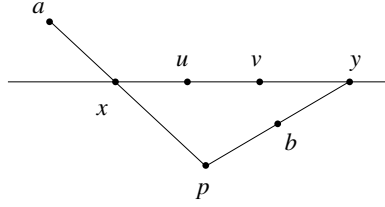


Fig. 3. Axiom A4.5.

whether \mathcal{D} , A3.3, A3.7 and **IP** imply **OP** (as claimed by Martin Schröder in his *Nachlass*). To get from $\{A2.2, A2.3, A2.5, A2.6, A2.11, A2.18, A3.3\}$ and **IP** to the Pasch axiom (i.e. to A3.1) one needs to add, either: (i) as shown in [246], the following axiom (Fig. 2)

A 4.2. $(\forall abxy)(\exists u)[(L(abu) \wedge L(xyu)) \vee (Z(aux) \wedge Z(buy)) \vee (Z(auy) \wedge Z(bux))]$,

or else (ii) as shown in [245], the axioms (Fig. 3)

A 4.3. $\bigwedge_{i=1}^3 \neg L(uva_i) \wedge S(a_1a_2uv) \wedge S(a_2a_3uv) \rightarrow S(a_1a_3uv)$,

A 4.4. $\neg L(uva) \wedge S(abuv) \wedge Z(acb) \rightarrow S(acuv)$,

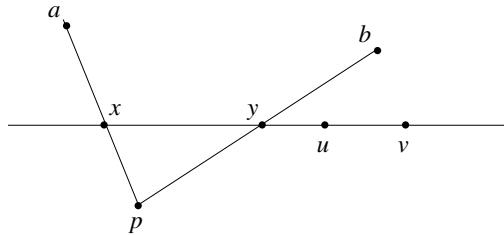
A 4.5. $(\forall abuv)(\exists pxy) [\neg L(uva) \wedge \neg L(uvb) \wedge \neg S(abuv) \rightarrow Z(pxa) \wedge Z(pby) \wedge L(uvx) \wedge L(uvy)]$,

where $S(abuv)$, to be used only if $u \neq v$ and to be read as ‘ a and b are on the same side of the line uv ’, is defined by (Fig. 4)

$$S(abuv) : \Leftrightarrow (\exists pxy) L(uvx) \wedge L(uvy) \wedge Z(pxa) \wedge Z(pyb).$$

These splittings of the Pasch axiom into **IP** and one more or three more axioms thus lead to alternative axiomatizations of ordered planar domains and of ordered planes.

Veblen’s [342] axiom system for ordered spaces (i.e. the axiom system obtained from his axiom system by removing the axiom fixing the dimension and the axiom of continuity), a theory we will

Fig. 4. The definition of $S(abuv)$.

refer to as Ω , has been simplified in [338]. The axiom system consists of {A2.2, A2.3, A4.6, A4.7, A2.18', A3.2, A2.38}, where A2.18' stands for A2.18 with $x \neq a$ added to the conclusion, and A4.6 and A4.7 stand for

$$\mathbf{A4.6.} \quad (\forall ab)(\exists x) a \neq b \rightarrow \neg (Z(abx) \vee Z(bxa) \vee Z(xab) \vee x = a \vee x = b),$$

$$\mathbf{A4.7.} \quad [((Z(abc) \vee Z(bca) \vee Z(cab)) \wedge Z(abd)) \vee (Z(bca) \wedge (Z(bda) \vee Z(dab)))] \rightarrow (Z(acd) \vee Z(cda) \vee Z(dac) \vee a = c \vee c = d \vee d = a).$$

Research regarding the at least three-dimensional models of Ω goes back to [5], where it was proved that the Desargues theorem must hold in Ω , as well as that a calculus of addition and multiplication can be defined for the points of an open segment in models of Ω , and that these operations satisfy certain rather weak properties. The representation theorem for models of Ω of dimension ≥ 3 , which generalizes the result of Sperner [308] for Desarguesian two-dimensional models of Ω , was proved in [70: p. 694]. It states that any at least three-dimensional model \mathfrak{M} of Ω can be embedded in a projective ordered space \mathfrak{P} , such that the separation relation defined in \mathfrak{M} by the betweenness relation of \mathfrak{M} by means of (4) is a restriction of the separation relation of \mathfrak{P} , and such that the image in \mathfrak{P} of \mathfrak{M} under the embedding is a semi-convex subspace of \mathfrak{P} . A presentation of a different proof of this result, using intricate projective methods, in particular the bundle theorem, inspired by [247] and citing [291, 30, 272, 276] as forerunners, can be found in [307] (another forerunner, not cited, is [201, 202] (an English summary of which can be found in [203: pp. 148–151, 204])). The result was generalized to ordered spaces that do not need to satisfy A2.18, i.e. where lines need to be only dense, but not unending, in [172: §5, 60: Ths. 3–5, Chap. VII].

A different presentation (not an axiomatization in the logical sense) of ordered spaces, based on *points* and a binary operation of *join*, with points as arguments and *segments* (regarded as subsets of the set of points), goes back to [264–266]. It was also treated in [16–18], and then in [49, 47, 34, 99]. That the convexity spaces defined in [49] are the same as our ordered spaces was sketched in [70: p. 684] (a complete proof was provided in [262] and repeated in [203]). Yet another approach can be found in [168]. Spaces endowed with a betweenness relation satisfying only Σ , but no Pasch-like axiom, have been considered in [57].

The question regarding weak axiom systems for ordered geometry in which classical results of convex geometry, such as the Radon, Carathéodory, Helly theorems, would hold, was first treated in [48], and then, very comprehensively in [59], where it was shown that a great part of convex geometry remains valid inside a theory axiomatized by an axiom system for Σ , as well as A3.2 and A4.1. That a generalized pigeonhole principle is all the sets involved in Helly's theorem need to satisfy, was shown in [76]. In [237] it was shown that one can also prove inside $\Sigma \cup \{A3.2, A4.1\}$ the 1979 Putnam problem mentioned in the Introduction.

5. Sperner's half-ordered geometry

5.1. Hyperplane-separation based half-ordered geometries

A very general ordered geometry, in terms of *points*, *hyperplanes*, and an *ordering function* assigning to each pair consisting of a point and a hyperplane one of the values 0, 1, or -1 , which generalizes both ordered spaces and ordered projective spaces, has been proposed by Sperner in [309] (see also [157]). The logical language for the two-dimensional case (on which we will focus in the sequel for the sake of simplicity), in which it can be expressed, will have to differ in several respects from the language in which Pasch's ordered geometry can be expressed. First, it will have to be a two-sorted language with variables for *points* (to be represented by lower-case Latin characters) and for *lines* (to be represented by lower-case Gothic characters). Second, it will have to contain two relation symbols, I , with $I(ag)$ to be read as 'point a is incident with line g ', and D , with $D(agb)$ to be read as 'the points a and b lie on different sides of line g '. The axioms for this theory of half-ordered planes are:

$$\mathbf{A5.1.} \quad (\forall ab)(\exists =^1 g) [a \neq b \rightarrow I(ag) \wedge I(bg)],$$

$$\mathbf{A5.2.} \quad (\exists abc)(\forall g) [\neg(I(ag) \wedge I(bg) \wedge I(cg))],$$

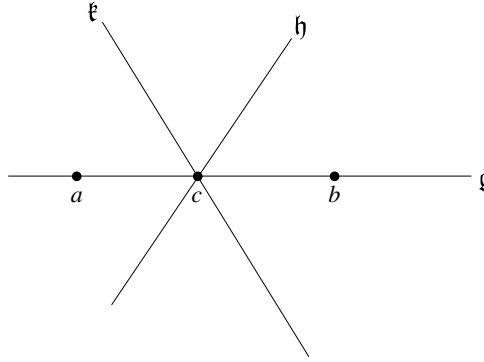


Fig. 5. Axiom A5.7.

A 5.3. $D(agb) \rightarrow \neg I(ag)$,

A 5.4. $D(agb) \rightarrow D(bga)$,

A 5.5. $\neg I(cg) \wedge D(agb) \rightarrow (D(agc) \vee D(bgc))$,

A 5.6. $\neg(D(agb) \wedge D(bgc) \wedge D(cga))$,

A 5.7. $a \neq b \wedge b \neq c \wedge c \neq a \wedge I(ag) \wedge I(bg) \wedge I(cg) \wedge h \neq g \wedge k \neq g \wedge I(ch) \wedge I(ck) \wedge D(ahb) \rightarrow D(akb)$.

A5.5 is a weak variant of Pasch's axiom, stating that if a line g does not pass through any of the points a , b , and c , and a and b are on different sides of g then so are at least one of the pairs $\{a, c\}$ and $\{b, c\}$. A5.6 is a variant of Pasch's theorem, stating that a line cannot separate all three pairs $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$. One of its special cases, when $a = b = c$, implies that a and b can be on different sides of g only if $a \neq b$. A5.7 is the so-called *Geradenrelation*, which states that, if g , h , and k are different lines meeting in c , and a and b , two points on g , are on different sides of h then they are on different sides of k as well (Fig. 5). Inside this theory, one can define, thanks to the presence of A5.7, which makes the definition independent of the choice of the line h , the betweenness notion Z by

$$Z(abc) :\Leftrightarrow (\exists gh) h \neq g \wedge I(ag) \wedge I(bg) \wedge I(cg) \wedge I(bh) \wedge D(ahc). \quad (10)$$

However, one cannot express the axioms of Sperner's [309] theory of half-ordered planes in terms of points and Z alone. Even trying to express the theory in terms of points alone is possible only if the two half-planes defined by every line are non-empty (in which case, we can rephrase the axiom system A5.1–A5.7 in terms of *points* and the quaternary relation v , with $v(aubv)$ standing for ' a and b are on different sides of the line uv ', and define the collinearity of three different points a , b , and c by asking that $v(cabx)$ not hold for any x). As one cannot define the fact that three points a , b , c are incident with a line g in terms of Z alone, given that there may well be lines on which there are three points a , b , c such that none is between the other two⁷ one cannot express the axioms in a language with Z alone, even if the two sides of every line are non-empty. One can also not express the relation D in terms of Z and I . In the special case of weakly ordered planes, studied in [245], one can express the geometry, although it is a half-ordered one, in terms of Z , for in it every triple of points of a line is, in some order, in a strict betweenness relation, and being on the same or on different sides of a line can be expressed in terms of Z (as in Figs. 3 and 4). As shown in [22], Z alone can also be used to axiomatize half-ordered affine planes.

⁷ Incidence can be expressed in terms of Z in some partially ordered geometries in logics extending first-order logic, such as the infinitary logic $L_{\omega_1\omega}$, as done in [263: p. 445], by stating that a , b , c are collinear if and only if there is a finite sequence of points, starting with a and b , and ending with c , such that each three consecutive terms of the sequence are, in some order, such that one of the terms lies between the other two.

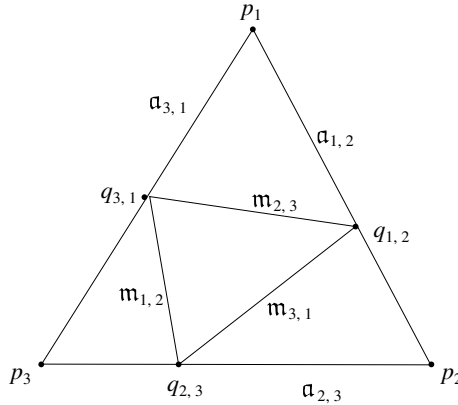


Fig. 6. Axiom A5.9.

Of the three axioms among A2.2–A2.12, involving three variables, the Z relation defined by (10) satisfies A2.2 and A2.4, but not A2.3. It satisfies none of the axioms A2.5–A2.12, but satisfies instead the following weaker axiom, known in the literature on half-ordered planes as the *Kürzregel*:

$$\mathbf{A\,5.8.} \quad \bigwedge_{i=1}^4 I(a_i g) \wedge \bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j \\ \rightarrow \left(\begin{array}{c} \forall \epsilon_k \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 \equiv 0 \pmod{2} \end{array} \quad \epsilon^1 Z(a_2 a_1 a_3) \wedge \epsilon^2 Z(a_3 a_1 a_4) \wedge \epsilon^3 Z(a_2 a_1 a_4) \right).$$

Half-ordered planes generalize ordered projective planes as well, for, with $\delta(abgh) \leftrightarrow (D(agb) \wedge D(ahb)) \vee (\neg D(agb) \wedge \neg D(ahb))$, one can define the separation relation $//$ by

$$a_1 a_2 // a_3 a_4 : \Leftrightarrow (\exists gh\ell) \bigwedge_{i=1}^4 I(a_i g) \wedge \bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j \wedge I(a_1 h) \wedge I(a_2 \ell) \wedge h \neq g \wedge \ell \neq g \wedge \neg \delta(a_3 a_4 h \ell). \quad (11)$$

It was shown in [153] that in Moufang projective planes, in which lines are incident with at least 4 points, and which satisfy A5.3–A5.6, the *Geradenrelation* A5.7 follows from the following axiom, called *Dreiecksrelation* (with addition in the indices being modulo 3, see Fig. 6):

$$\mathbf{A\,5.9.} \quad \bigwedge_{i=1}^3 I(p_i a_{i,i+1}) \wedge I(p_{i+1} a_{i,i+1}) \wedge I(q_{i,i+1} a_{i,i+1}) \wedge I(q_{i,i+1} m_{i+1,i+2}) \wedge I(q_{i,i+1} m_{i-1,i}) \wedge p_i \neq q_{i,i+1} \\ \wedge p_{i+1} \neq q_{i,i+1} \rightarrow \left[\begin{array}{c} \forall \epsilon_i \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 \equiv 0 \pmod{2} \end{array} \quad \bigwedge_{i=1}^3 \epsilon^i D(p_i m_{i,i+1} p_{i+1}) \right].$$

All half-ordered planes satisfy the *Dreiecksrelation*. It is not known whether the condition that a projective plane be Moufang is needed for A5.9 to imply A5.7 in the presence of the other half-order axioms.

The effects of the half-order on the ternary field associated with a projective or an affine plane were comprehensively studied in [152,141,154].

That the half-order of half-ordered Desarguesian affine planes can be extended to a half-order of its projective extension, provided that the affine plane is not the minimal affine plane (of order 2), was shown in [311], and that this result holds for all translation planes whose kernel is different from $GF(2)$ and $GF(3)$ was shown in [179]. For affine planes that do not need to satisfy the major Desargues axiom, the problem was treated independently in [155,135], and it turns out that A5.10 and A5.11 are the necessary and sufficient requirements for the projective extendability of the half-order, i.e. of D (in a language with variables for both points and lines, and, in addition to I and D , a binary relation $||$ of line parallelism), where:

A 5.10. $a \neq b \wedge I(a\ell) \wedge I(b\ell) \wedge \ell \parallel g \wedge \ell \parallel h \wedge \ell \neq g \wedge \ell \neq h \rightarrow (D(agb) \leftrightarrow D(ahb)),$

$$\mathbf{A\ 5.11.} \left\{ \bigwedge_{i=1}^3 p_i \neq p_{i+1} \wedge I(p_i a_{i,i+1}) \wedge I(p_{i+1} a_{i,i+1}) \right. \\ \left. \rightarrow \left[\bigwedge_{i=1}^3 m_{i,i+1} \parallel a_{i,i+1} \wedge m_{i,i+1} \neq a_{i,i+1} \wedge \left[\bigvee_{\substack{\epsilon_j \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 \equiv 0 \pmod{2}}} \bigwedge_{j=1}^3 {}^{\epsilon_j} D(p_j m_{j,j+1} p_{j+1}) \right] \right] \right\}.$$

Examples of affine planes with half-order not extendible to one of the projective closure were provided in [135]. That the two sides determined by a line in an affine plane have the same number of elements if the half-order satisfies $(\forall g)(\exists a)(\exists b) D(agb)$ and $a \neq b \wedge I(a\ell) \wedge I(b\ell) \wedge \ell \parallel g \rightarrow \neg D(agb)$ was shown in [97].

The connection between the geometric notion of ordering function and a very weak notion of ordering of the skew field coordinatizing a projective plane has been studied ever since the introduction of ordering functions in [309,311]. Let K be a skew field and $P \subset K \setminus \{0\}$, such that we have $a \cdot b \in P$ precisely when both a and b are either in P or in $K \setminus (\{0\} \cup P)$. The elements of P will be referred to as *positive* and those of $K \setminus (\{0\} \cup P)$ as *negative*, and the pair $\langle K, P \rangle$ a half-ordered field. If we denote by $\mathfrak{P}_2(K)$ the projective plane over the skew field K , whose points are right-homogeneous triples $\mathbf{x} = (x_1, x_2, x_3)$ (i.e. (x_1, x_2, x_3) and (y_1, y_2, y_3) represent the same point if and only if $y_i = x_i \cdot \lambda$, with $\lambda \in K \setminus \{0\}$) and lines are left-homogeneous triples $\mathbf{u} = [u_1, u_2, u_3]$ (i.e. $[u_1, u_2, u_3]$ and $[v_1, v_2, v_3]$ represent the same line if and only if $v_i = \lambda \cdot u_i$, with $\lambda \in K \setminus \{0\}$), with the x_i (respectively u_i) not all zero, with the point \mathbf{x} being incident with line \mathbf{u} if and only if $\mathbf{u} \cdot \mathbf{x} := u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$, we can define $\delta(\mathbf{a}\mathbf{b}\mathbf{g}\mathbf{h})$ to hold if and only if the number of negative elements among $\mathbf{g} \cdot \mathbf{a}$, $\mathbf{g} \cdot \mathbf{b}$, $\mathbf{h} \cdot \mathbf{a}$, and $\mathbf{h} \cdot \mathbf{b}$ is even, and in terms of δ , the separation relation $//$ as in (11). By picking, for every point \mathbf{a} (for every line \mathbf{g}) of the projective plane a particular representative $\mathbf{x}_\mathbf{a}$ ($\mathbf{u}_\mathbf{g}$) from the class of right-homogeneous (left-homogeneous) triples representing \mathbf{a} (\mathbf{g}), we can also define D by stipulating that

$$D(\mathbf{a}\mathbf{g}\mathbf{b}) \text{ holds precisely when } (\mathbf{u}_\mathbf{g} \cdot \mathbf{x}_\mathbf{a}) \cdot (\mathbf{u}_\mathbf{g} \cdot \mathbf{x}_\mathbf{b}) \text{ is negative,} \quad (12)$$

to obtain a half-ordered plane. Although, as shown in [311], not all half-ordered projective planes can be obtained from half-ordered skew fields by means of (12), all half-ordered Desarguesian projective planes can be coordinatized by half-ordered skew-fields, by stipulating that $c \in K \setminus \{0\}$ is in P if and only if $\delta((1, 1, 0)(1, c, 0)[1, 0, 0][0, 1, 0])$ holds, where δ is defined in terms of D .

A geometric characterization – in a language extending that of point, line, I and D by two binary operations, the first one with points and the second one with lines as variables and values – for those projective planes satisfying A5.3–A5.6, whose plane separation predicate D can be defined as in (12), where the positive cone of the underlying skew field does not need to satisfy any multiplicative closure condition, can be found in [151].

The most surprising and difficult to prove result to come out of Sperner's theory of half-ordered planes is Jousen's characterization of half-ordered planes which can be embedded in ordered projective planes. Jousen [136], continuing work started in [134], showed that any model \mathfrak{M} of {A5.1–A5.6, A5.12, A5.13} can be embedded in a projective ordered plane \mathfrak{P} , whose separation relation $//_\mathfrak{P}$ is an extension of the separation relation $//_\mathfrak{M}$, defined in \mathfrak{M} terms of $I_\mathfrak{M}$ and $D_\mathfrak{M}$ by (11). The axioms A5.12 and A5.13 are:

A 5.12. $(\forall \mathbf{g})(\exists a_1 a_2 a_3 a_4) [\bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j \wedge \bigwedge_{i=1}^4 I(a_i \mathbf{g})],$

$$\mathbf{A\ 5.13.} [\bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j \wedge h_i \neq h_j \wedge \bigwedge_{i=1}^4 I(a_i h_i) \wedge ((\bigwedge_{i=1}^4 I(a_i \mathbf{g})) \vee (\bigwedge_{i=1}^4 I(o h_i)))] \\ \rightarrow \left[\bigvee_{\substack{\epsilon_i \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 2}} {}^{\epsilon_1} \delta(a_3 a_4 h_1 h_2) \wedge {}^{\epsilon_2} \delta(a_2 a_4 h_1 h_3) \wedge {}^{\epsilon_3} \delta(a_2 a_3 h_1 h_4) \right].$$

One part of A5.13 (the one corresponding to the $\bigwedge_{i=1}^4 I(a_i g)$ disjunct) states that, if a_1, a_2, a_3, a_4 are four different collinear points, then exactly one of the separation relations $a_1 a_2 / a_3 a_4, a_1 a_3 / a_2 a_4, a_1 a_4 / a_2 a_3$ holds. Its other part (corresponding to the $\bigwedge_{i=1}^4 I(o h_i)$ disjunct) is the dual statement (in the sense of projective geometry).

The theory of half-orders has been extended to the so-called *Benz planes* (with Möbius, Laguerre and Minkowski planes as special cases, see [38,50]), which are incidence-based geometries, with *points* and *cycles* as individual variables, by Kroll in [174–177], as well as in [9–11,158,159].

5.2. Half-ordered planes based on strict betweenness and incidence

Some half-ordered planes, to be referred to as *Paschian half-ordered planes*, can be expressed in a point-and-line-based language, with I and Z as predicate symbols. Their axiom system, in [171], consists of A5.1, A5.2, A5.14, A5.8, and the axiom A3.1_σ expressed in terms of points, lines, incidence, and strict betweenness, where

A 5.14. $(\forall g)(\exists a_1 a_2 a_3) [\bigwedge_{i=1}^3 a_i \neq a_{i+1} \wedge I(a_i g)]$.

As shown in [169: Satz 1], A5.8 can be replaced in the axiom system for Paschian half-ordered planes by A2.3 and A2.13.

The results proved in [171] involve Paschian half-ordered punctured projective planes (those missing one point), analyze the effect the axiom, stating that from every point to every line there are only finitely many parallels, has on Paschian half-ordered planes, as well as to non-Fanoian affine planes (there exist parallelograms with parallel diagonals), which, if Paschian half-ordered, must be such that, for any three distinct points a, b , and c on any line, an even number of relations $Z(abc)$, $Z(bca)$, $Z(cab)$ must hold. In the same vein are the results in [173]. A like-minded result in [161]: If to dimension-free incidence spaces (in which the dimension ≥ 2 is not fixed), axiomatized in terms of points, lines, and planes, with the respective incidence relations, one adds an axiom corresponding to A2.20, as well as A2.2, A2.18, and the strong form of the Pasch axiom in which the hypothesis also states that the transversal line is in the plane of the triangle, then the parity of the number of the relations $Z(abc)$, $Z(bca)$, $Z(cab)$ which hold is the same for all collinear triples (a, b, c) . Variations on this theme are in [191].

The setting of Paschian half-ordered planes also allows the most general treatment, to be found in [132], of the question: When are segments convex? This problem had been dealt with in the context of Desarguesian affine planes in [310] and without the Desarguesian requirement in [156]. The half-order of a Paschian half-ordered plane is called *almost trivial*, if the plane is such that, for all collinear triples (a, b, c) , either none of or exactly two of the relations $Z(abc)$, $Z(bca)$, $Z(cab)$ hold, and it also satisfies:

A5.15. $a_1 \neq a_2 \wedge \bigwedge_{i,j=1}^2 x_i \neq a_j \wedge I(a_i g) \wedge I(x_i g) \rightarrow \neg(Z(a_1 x_1 a_2) \wedge \neg Z(a_1 x_2 a_2))$.

As shown in [132], if a Paschian half-ordered plane satisfies A2.7, then its segments are convex, and it is either (i) an almost trivial plane, or (ii) the affine plane of order 3, or (iii) an ordered planar domain (i.e., of three points on a line, one and only one lies between the other two, and thus Z satisfies the axioms A2.2–A2.12).

5.3. Ordered planar domains based on incidence and line-separation

As shown in ([163: p. 82, 83]), ordered planar domains can also be axiomatized in the same language as half-ordered planes, by adding to the axioms of half-ordered planes A5.14 and the following two axioms

A 5.16. $\bigwedge_{i=1}^3 I(a_i g) \wedge I(a_i a_i) \wedge a_i \neq a_{i+1} \rightarrow \left[\bigvee_{\substack{\epsilon_j \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 1}} \bigwedge_{i=1}^3 \epsilon_j D(a_i a_{i+1} a_{i+2}) \right],$

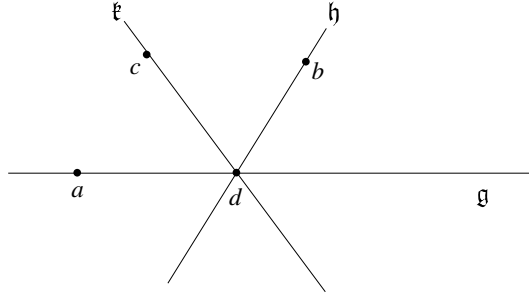


Fig. 7. Axiom A5.18.

A 5.17. $(\forall abgh)(\exists p) [I(ah) \wedge I(bh) \wedge D(agb) \rightarrow I(ph) \wedge I(pg)]$.

It is in this language that Jaritz [131] proves that ordered planar domains must satisfy the following form of the Fano axiom (Fig. 7),

A 5.18. $I(ag) \wedge I(dg) \wedge I(bh) \wedge I(dh) \wedge I(de) \wedge I(ce) \wedge a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d \wedge \neg D(bgc) \wedge \neg D(ahc) \rightarrow D(aeb)$,

and thus also the projective form of the Fano axiom (the diagonal points of a complete quadrangle are not collinear). In ordered affine planes, A5.18 implies ([131: Satz 4]) the affine form of Fano's axiom (the diagonals of no parallelogram are parallel). That the projective form of the Fano axiom holds in ordered projective planes had been shown earlier in [180].

6. Pre-orders, quasi-orders, and semi-orders

While Sperner's half-orders correspond, in the case of commutative fields, to half-orderings of the underlying field (the positive cone being closed under multiplication, but not necessarily under addition), Szczerba and Szmielew [318], in their investigation on the Cartesian planes corresponding to Pasch-free Euclidean geometry (a geometry discovered in [315]), have stumbled upon semi-ordered fields, in which the positive cone is closed under addition but not necessarily under multiplication (a summary of the literature on the subject can be found in [321]). Tecklenburg [326–332] has on the one hand generalized the semi-orders found in [318] to semi-orders of affine planes, and has weakened the notion of semi-order to that of a quasi-order and to that of a pre-order, which are weakenings of half-orders as well. She has also thoroughly studied the corresponding pre-ordered, quasi-ordered, and semi-ordered algebraic structures appearing in the Lenz–Barlotti classification. The impressive results were collected in the comprehensive monograph [333].

7. Other ordered incidence geometries

Ordered affine and projective Hjelmslev and Klingenberg planes were studied in [26–29,182,205–208,210,211].

8. Archimedean orders

A discussion of the Archimedean nature of an ordered plane or space seems at first to not belong to a survey of first-order axiomatizations of ordered incidence spaces, given that Archimedeanity seems to require the metric concept of length, and because it was plain, at least since the Löwenheim–Skolem theorem (a first-order theory with infinite models must have models in all infinite cardinalities), that the Archimedean property cannot be captured by first-order logic.

While the first argument for its exclusion can be bypassed by formulating the Archimedean axiom in a manner that does not refer to lengths of segments, the second one forces us to go beyond first-order logic for the sake of this axiom, to logics which are, nevertheless, significantly weaker than second-order logic. The modern history of the Archimedean axiom was surveyed in [79,77,78].

The Archimedean nature of the order of an affine ordered plane (with no significant difference in higher dimensions, so it is enough to focus on the two-dimensional case) can be expressed in terms of a notion of point-reflection, which, in general, i.e. unless the affine plane is a translation plane, depends on the auxiliary points used to define it.

With $\pi(abcd)$, to be read ‘ a, b, c, d are the vertices of a parallelogram, such that $ab \parallel cd$ and $ac \parallel bd$ ’, $xy \simeq_c ab$ to be read as ‘ ab and xy are two collinear segments of the same “length” and of the same orientation, with respect to c ’, and $\sigma_b(axx')$ to be read as ‘ x' is the reflection, using the auxiliary point b , of x in a ’, we have

$$\begin{aligned}\pi(abcd) &:\Leftrightarrow ab \parallel cd \wedge ac \parallel bd, \\ xy \simeq_c ab &:\Leftrightarrow (\exists d) \neg L(abc) \wedge \pi(abcd) \wedge \pi(xycd) \wedge L(abx), \\ xy \simeq ab &:\Leftrightarrow (\exists c) xy \simeq_c ab, \\ \sigma_b(axx') &:\Leftrightarrow xa \simeq_b ax',\end{aligned}$$

and the Archimedean axiom stipulates that, given any three points a, b , and c , with $Z(abc)$, one of the terms of the sequence $\{b_n\}_{n \in \mathbb{N}}$ – defined by using an auxiliary point d with $\neg L(abd)$, recursively by $b_{-1} := a$, $b_0 := b$, and $\sigma_d(b_i b_{i-1} b_{i+1})$ for $i \geq 0$ – say b_m , must be such that $Z(acb_m)$.

This can be expressed in several extensions of first order logic. The options are:

- (i) a fragment of the infinitary logic $L_{\omega_1 \omega}$, called *algorithmic logic* containing only Boolean combinations of halting-formulas for flow-charts (that may contain loops but not recursive calls). This logic, in which no quantification is allowed, was introduced by Engeler [81], and its relevance to geometry was studied in [82–85,242] and [300] (where the affine context was first considered). It turns out that the only fact that algorithmic logic – based on Z and a parallelogram operation, producing the fourth vertex of a parallelogram, whenever the first three are given – is noticing to hold in the ordered affine plane over the real numbers, that is not already a sentence of first-order logic, is the affine form of the Archimedean axiom.

Expressed as

$$Z(b_{-1} b_0 c) \rightarrow \left[\bigvee_{n=0}^{\infty} \bigwedge_{i=0}^n \sigma_d(b_i b_{i-1} b_{i+1}) \wedge Z(b_{-1} c b_{n+1}) \right]$$

the affine Archimedean axiom is indeed the halting-formula for a flow chart, namely the one with b_{-1} , b_0 , and c as inputs, asking the question whether $Z(b_{-1} c b_{n+1})$ holds or not, stops if it holds, and goes on to ask the same question with $n+2$ instead of $n+1$, otherwise.

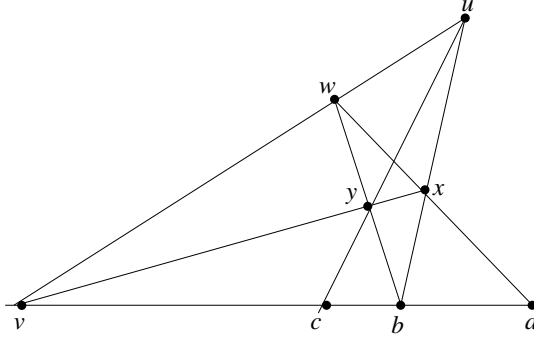
- (ii) weak second-order logic, $L(II_0)$, in which, in addition to the individual variables, there are variables to be interpreted as finite sets of individual variables, and a relation \in , with $x \in X$ having its customary set theoretic meaning (see [292] for a precise definition and for properties of this logic). In it, the affine form of the Archimedean axiom can be expressed as

$$\begin{aligned}(\forall abcd)(\exists X)(\exists up)(\forall t)(\exists t') \{ &Z(abc) \wedge \neg L(abd) \\ \rightarrow [&u \in X \wedge \sigma_d(bau) \wedge p \in X \wedge Z(ACP) \wedge (t \in X \wedge t \neq p \rightarrow t' \in X \wedge ab \simeq_d tt')]\}.\end{aligned}$$

- (iii) $L(Q^2)$, logic with the Ramsey quantifier Q^2 , a new quantifier binding two variables, which extends first-order logic by adding $Q^2 xy \varphi$, where x and y are variables, and φ is a formula with x and y free, to the list of well-formed formulas, the intended interpretation of $Q^2 xy \varphi$ being ‘there is an infinite set I , such that, whenever x and y are interpreted as distinct elements a_1 and a_2 of I , the formula φ holds’. In $L(Q^2)$, the affine form of the Archimedean axiom is the negation of (see [60])

$$\begin{aligned}(\exists abcd)(Q^2 xy)(\exists uv) [&Z(abc) \wedge \neg Z(abd) \wedge ab \simeq_d xu \wedge ab \simeq_d yv \wedge (Z(xuy) \vee u = y \\ &\vee \vee Z(yvx) \vee v = x) \wedge Z(byc) \wedge Z(bxc)].\end{aligned}$$

- (iv) $L(DTC)$, a weak form of *deterministic transitive closure logic* (see. [75: 8.6]), which extends first-order logic by adding $[DTC_{x,y} \varphi]st$, where x and y are variables and s and t are terms, to the

Fig. 8. The definition of $\theta(abcuvw)$.

list of well-formed formulas. The meaning of $[DTC_{x,y}\varphi]st$ in a model \mathfrak{M} with universe M is that $(s, t) \in DTC(\{(x, y) \in M^2 \mid \mathfrak{M} \models \varphi(x, y)\})$, where $DTC(\{(x, y) \in M^2 \mid \mathfrak{M} \models \varphi(x, y)\})$ stands for the deterministic transitive closure of the set in parentheses, i.e. the set of couples $\{(a, b) \in M^2 \mid \text{there exist } n \geq 1 \text{ and } e_0, \dots, e_n \in M \text{ such that } a = e_0, b = e_n, \text{ and for all } i < n, e_{i+1} \text{ is the unique } e \text{ for which } \mathfrak{M} \models \varphi(e_i, e)\}$. In $L(DTC)$, the affine form of the Archimedean axiom can be expressed as

$$(\forall abc)(\exists d) [Z(abc) \rightarrow [DTC_{x,y}xy \simeq ab]bd \wedge Z(acd)].$$

According to [61,127], $L(DTC)$ is the weaker than both $L(II_0)$ and $L_{\omega_1\omega}$, and $L(Q^2)$ is not comparable with (neither weaker nor stronger than) any of $L(II_0)$, $L_{\omega_1\omega}$, or $L(DTC)$.

One of the most significant results the Archimedeanity of the order of a Desarguesian affine plane implies is the Pappus axiom, in other words, the commutativity of the multiplication of the coordinatizing ordered skew field. That a much weaker, first-order version of the Archimedean axiom suffices to prove that a Desarguesian ordered affine plane is Pappian was shown in [273,275]. With $ab \simeq cd$ being defined in terms of the midpoint operation μ , $\mu(ab)$ standing for the midpoint of the segment ab , by $ab \simeq cd : \Leftrightarrow \mu(ad) = \mu(bc)$, the first order version of the affine Archimedean axiom reads as the following axiom schema:

$$(\exists pq)(\forall rst)(\exists uv) [p \neq q \wedge \varphi(p) \wedge \varphi(q) \wedge (pq \simeq rs \rightarrow (\varphi(r) \Leftrightarrow \varphi(s))) \rightarrow (L(tpq) \rightarrow \varphi(u) \wedge \varphi(v) \wedge Z(utv))],$$

where φ is a formula containing none of the variables p, q, r, s, t, u, v . This axiom schema states that, if \mathfrak{A} is a definable set of points (defined by φ), containing the distinct points \mathbf{p} and \mathbf{q} , and such that, whenever $\mathbf{ab} \simeq \mathbf{cd}$, \mathbf{c} is in \mathfrak{A} if and only if \mathbf{d} is in \mathfrak{A} , then every point \mathbf{t} on the line \mathbf{pq} is between two points of \mathfrak{A} . This is far from being the only set of first-order sentences true in all Archimedean ordered Pappian affine planes. Indeed, as shown by Rautenberg [274], the first-order theory of the class of Archimedean ordered Pappian affine planes is not recursively axiomatizable.

For the case of projective planes, one may stipulate that an ordered projective plane is Archimedean precisely if all affine planes resulting from the removal of a line are Archimedean (for all possible choices of the point c used to define $xy \simeq_c ab$).

One can also state the Archimedean axiom for the most general case, namely that of ordered spaces, for which we first introduce the following defined predicate (Fig. 8):

$$\theta(abcuvw) : \Leftrightarrow (\exists xy) [Z(bxu) \wedge Z(axw) \wedge Z(vyx) \wedge Z(byw) \wedge Z(uyx) \wedge Z(bc v)].$$

If we think of the line vu as the line at infinity and assume that we have $Z(abv)$ as well (which will be the case in the instance in which we will use θ), then $\theta(abcuvw)$ stands for the fact that the point c is the reflection of a in b , and $\theta(abcuvw)$ can thus be thought of as asserting that c is (a projective geometry view of) the ‘reflection’ of a in b , constructed with the help of u, v, w, x, y .

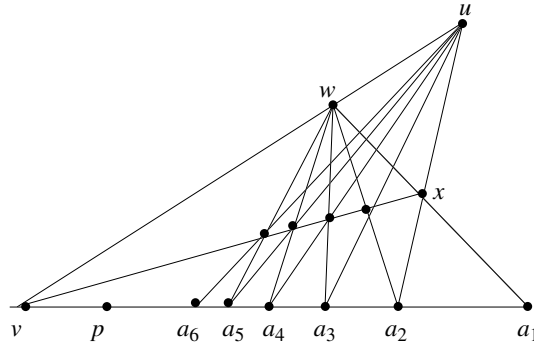


Fig. 9. The Archimedean axiom.

The Archimedean axiom can be expressed for ordered spaces in the following form: Given two points a_1 and a_2 , a point p on the ray $\overrightarrow{a_1 a_2}$, and a line uw (which we may think of as the ‘line at infinity’) meeting the ray $\overrightarrow{a_1 a_2}$ in a point v (which we may think of as ‘at infinity’), which is such that p is strictly between a_2 and v , the sequence of points a_i , obtained by iterating the ‘reflection’ operation, first ‘reflecting’ a_1 in a_2 to get a_3 (by means of $\theta(a_1 a_2 a_3 uvw)$), then a_2 in a_3 to get a_4 , and so on, will eventually move past p , i.e., for some n , we will find p lying between a_2 and a_{n+2} (Fig. 9).

$$\begin{aligned} & \{-L(a_1 v w) \wedge Z(a_1 x w) \wedge Z(a_2 x u) \wedge Z(v w u) \wedge Z(a_1 a_2 v) \wedge Z(a_2 p v) \\ & \rightarrow [\bigvee_{n=1}^{\infty} ((\exists a_3 \dots a_{n+2}) \wedge \bigwedge_{i=1}^n \theta(a_i a_{i+1} a_{i+2} uvw) \wedge B(a_2 p a_{n+2}))]\}. \end{aligned}$$

A remarkable result, proved by Prieß-Crampe – after preliminary work in [65,269] – in [270] (see also [271], in which one can find a good survey of all results on Archimedean ordered projective planes up to 1983), states that every ordered projective Archimedean plane can be embedded into a topological projective plane the point space of which is a surface. The latter are called flat projective planes, and have been thoroughly studied, see [284: Ch. 3]. A consequence of this result and of other results by Salzmann is that all Archimedean ordered projective planes of the Lenz class III must be Moulton planes (a class of planes introduced in [232]), i.e. are subplanes of the real projective plane, in which the multiplication of the reals has been changed to \circ , with $a \circ b$ being defined as arb (r being a positive constant $\neq 1$) if both a and b are negative, and ab , otherwise.

As shown in [144], there are Archimedean ordered affine planes, for which the order of the projective extension is not Archimedean. Moreover, there are non-Archimedean ordered projective planes which contain an infinite number of Archimedean ordered affine planes. Given that, as shown in [137], free projective planes allow an Archimedean ordering, none of the traditional configuration theorems (no form of Desargues, for example) can be derived from the order and the Archimedean axioms alone. Various results on certain classes of Archimedean ordered projective planes were proved in [142,143,145,146,148,149].

In [238] it is shown that, in Archimedean ordered hyperbolic planes, the negated point-equality \neq is positively definable (i.e. with \wedge , \vee , and $\bigvee_{n=1}^{\infty}$ as the only sentential connectives allowed in the definiens) in $L_{\omega_1 \omega}$ in terms of B .

In [150] it is shown that \parallel (the relation defined in (9)) can be defined by means of a positive existential definition (the only quantifier allowed is \exists and the only sentential connectives allowed are \wedge , \vee , $\bigvee_{n=1}^{\infty}$) in terms of B inside the theory of Archimedean ordered affine planes.

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